1/28/Calc3 Quiz 3

$u = \langle -2, 6, -3 \rangle$

$v = \langle -3, 1, 1 \rangle$

(a) $\bar{u} \cdot \bar{v} = 6 + 6 - 3 = 9$

(b) $\cos\theta = \dfrac{\bar{u} \cdot \bar{v}}{|\bar{u}||v|} = \dfrac{9}{7\sqrt{11}}$

$\sqrt{4+36+9}$ $\qquad$ $\sqrt{9+1+1}$

(c) $w = \langle 1, 4, -1 \rangle$

$\bar{u} \cdot \bar{w} = 25 \neq 0$ $\qquad$ (not $\perp$)

$\bar{v} \cdot \bar{w} = 0$ $\qquad$ $v \perp w$

(d) $\text{Proj}_{\bar{v}} \bar{u} = \dfrac{\bar{u} \cdot \bar{v}}{|v|^2} v$

$\dfrac{9}{11} \langle -3, 1, 1 \rangle = \left\langle -\dfrac{27}{11}, \dfrac{9}{11}, \dfrac{9}{11} \right\rangle$

Last time

Parametric lines in $\mathbb{R}^3$:

$$L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \ t \in \mathbb{R}$$

point on L ($\bar{r}_0$) direction of L

$$\bar{r}(t) = \bar{r}_0 + \bar{v}\, t$$

$\bar{r}_0$: point, $\bar{v}$: direction

Ex1

(a) Find the line L_1 through $P = (1,2,4)$ and $R = (3,5,-1)$

direction: $\bar{v} = \overrightarrow{PR} = \langle 2, 3, -5 \rangle$

so L_1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}$

(b) Is $Q = (-5, -7, 19)$ on L_1?

i.e. find t:

$$-5 = 1 + 2t \rightsquigarrow t = -3$$
$$-7 = 2 + 3t \rightsquigarrow$$
$$19 = 4 - 5t$$

$t = -3$ works for $x/y/z$ coords,

$\therefore Q$ is on L_1

(c) Does L_1 meet the line

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3-4t \\ 7-6t \\ 1+10t \end{pmatrix}$$

<u>Observation</u> $L_1 \parallel L_2$:

$$\begin{pmatrix} -4 \\ -6 \\ 10 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$$S = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} \text{ on } L_2$$

Is $\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ on L_1 ?

$$L_1 : \begin{pmatrix} 1 + 2t \\ 2 + 3t \\ 4 - 5t \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} \quad \begin{matrix} t = 1 \\ t = 5/3 \end{matrix}$$

NO : $\therefore$ $L_1 // L_2$ but $L_1 \neq L_2$ so L_1, L_2 disjoint.

(d) Show that L_1 intersects

the line L_3

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+3t \\ 13-4t \\ -3+t \end{pmatrix} \quad t \in \mathbb{R}$$

$$L_1 : \begin{pmatrix} 1+2t \\ 2+3t \\ 3 - 5t \end{pmatrix}$$

$$= \begin{pmatrix} -3+3s \\ 13-4s \\ -3+s \end{pmatrix} \quad s \in \mathbb{R}$$

$$\left(\begin{array}{ll} x & 1+2t = -3+3s \\ y & 2+3t = 13-4s \\ z & 4-5t = -3+s \end{array} \right.$$

$$-3s + 2t = -4$$

$$\boxed{4s + 3t = 11}$$

$$-s - 5t = -7$$

$\times 4$

$$-4s - 20t = -28$$

add

$$0s - 17t = -17$$

Also solve x-equ

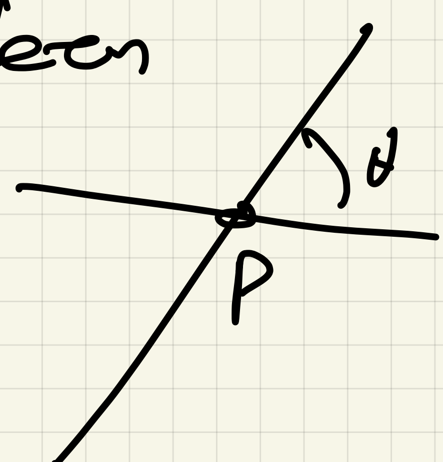
$$\boxed{t = 1}$$

$$\boxed{s = 2}$$

$$\Rightarrow P = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

(e) what is the angle of intersection between L_1 & L_2

smallest angle



directions of lines:

$\bar{u} = \langle 2, 3, -5 \rangle$

$\bar{v} = \langle 3, -4, 1 \rangle$

$6 - 12 - 5$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|u||v|} = \frac{-11}{\sqrt{38}\sqrt{26}}$$

$$\frac{-11}{\sqrt{38}\sqrt{26}} < 0 \Rightarrow \quad \theta > 90^\circ$$

to get smaller,

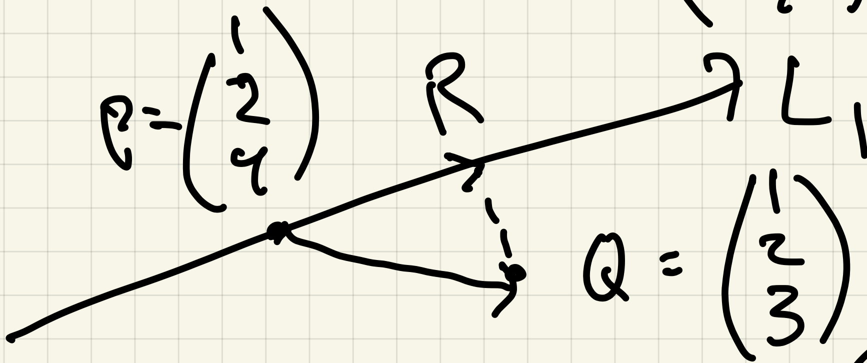
$$\cos\theta = \frac{11}{\sqrt{38}\sqrt{26}}$$

$$\Downarrow$$

$$\theta = 69.52^\circ \cong 1.213 \text{ rad}$$

Ex2 Find distance from point $Q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (1, 2, 3)$ to line

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}$$



(A) Solve as Calc l problem

$d(t)$ = dist from $\begin{pmatrix} 1+2t \\ 2+3t \\ 4-5t \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$d(t) = \sqrt{((1+2t)-1)^2 + \text{---}}$$

minimize $d(t)$:

(B) Calculate $\overrightarrow{PR} =$

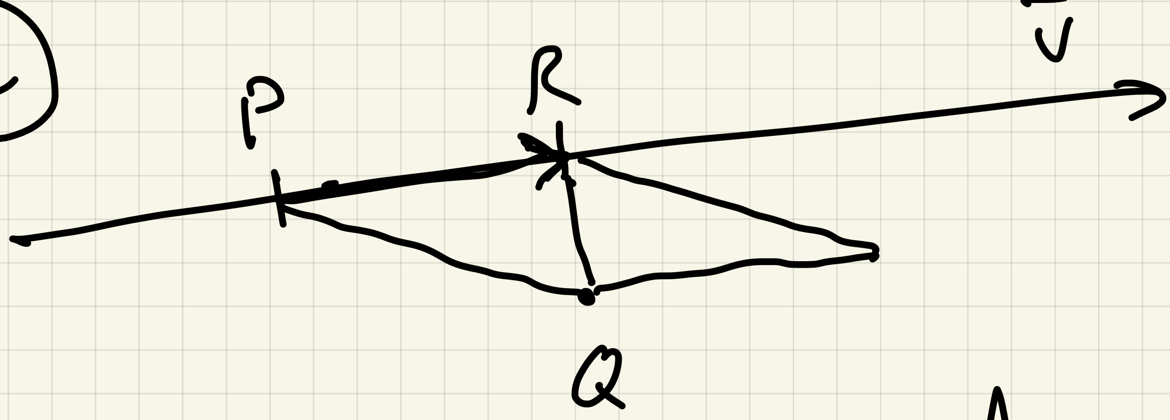
$$\mathrm{Proj}_{\vec{v}_{\parallel}} \overrightarrow{PQ}$$

$$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

Then $\vec{RQ} = \vec{PQ} - \vec{PR}$

dist $= |\vec{RQ}|$

(c)



$|RQ| = $ ht of $\square = \frac{\text{Area}}{\text{base}} =$

$$\frac{|PQ \times PR|}{|PR|}$$

$\vec{PR} = c\bar{v}$

$= c\langle 2,3,5\rangle$

$$\frac{|\overline{PQ} \times cv|}{|cv|} = \frac{|\cancel{c}|\,|\overline{PQ} \times v|}{\cancel{|c|}\,|v|} =$$

$$\frac{|\vec{PQ} \times \bar{v}|}{|v|} = \frac{\sqrt{13}}{\sqrt{38}} = \sqrt{\frac{13}{38}}$$

$$\overrightarrow{PQ} \times \bar{v} = \begin{vmatrix} \hat{i} & j & k \\ 0 & 0 & -1 \\ 2 & 3 & -5 \end{vmatrix} =$$

$$|\langle 3, -2, 0 \rangle| =$$

$$\sqrt{13}$$

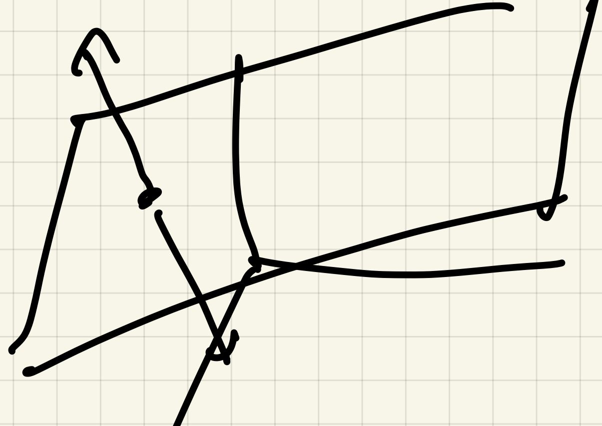
$$|v| = \sqrt{38}$$

so

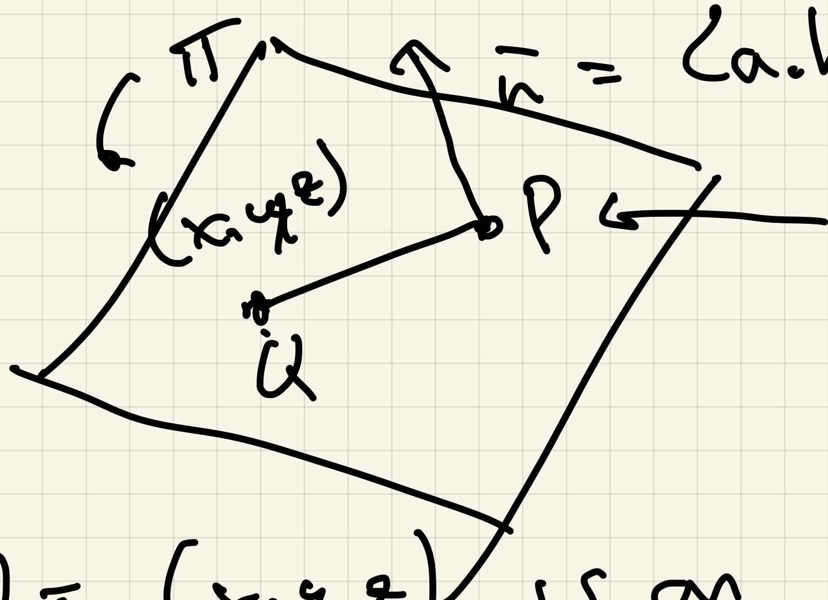
<u>In general</u>: the distance from point Q to the line $r(t) = \bar{P} + \bar{v}t$, is

$$\frac{|\overrightarrow{PQ} \times \bar{v}|}{|v|}$$

<u>Planes in $\mathbb{R}^3$</u>:



A plane has <u>no</u> direction,
but does have a <u>normal vector</u>
or <u>perpendicular</u>



$\bar{n} = \langle a, b, c\rangle$ normal vector

$P = (x_0, y_0, z_0)$

$Q = (x, y, z)$ is on plane Π

$$\Updownarrow$$

$$\overrightarrow{PQ} \perp \bar{n} \iff \overrightarrow{PQ} \cdot \bar{n} = 0$$

$$\iff (x - x_0, y - y_0, z - z_0) \cdot \langle a, b, c\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

standard form of plane equation

Ex) The plane through

$P = (3,5,-1)$ normal to

$\bar{n} = \langle 2,3,5 \rangle$ has equation

$$2(x-3)+3(y-5)+5(z+1)=0$$

Simplified form

$$2x+3y+5z=16$$

$$-6 \quad -15 \quad +5$$

Note: ① Read off normal direction

$\langle 2,3,5 \rangle$

③ Can sketch picture:

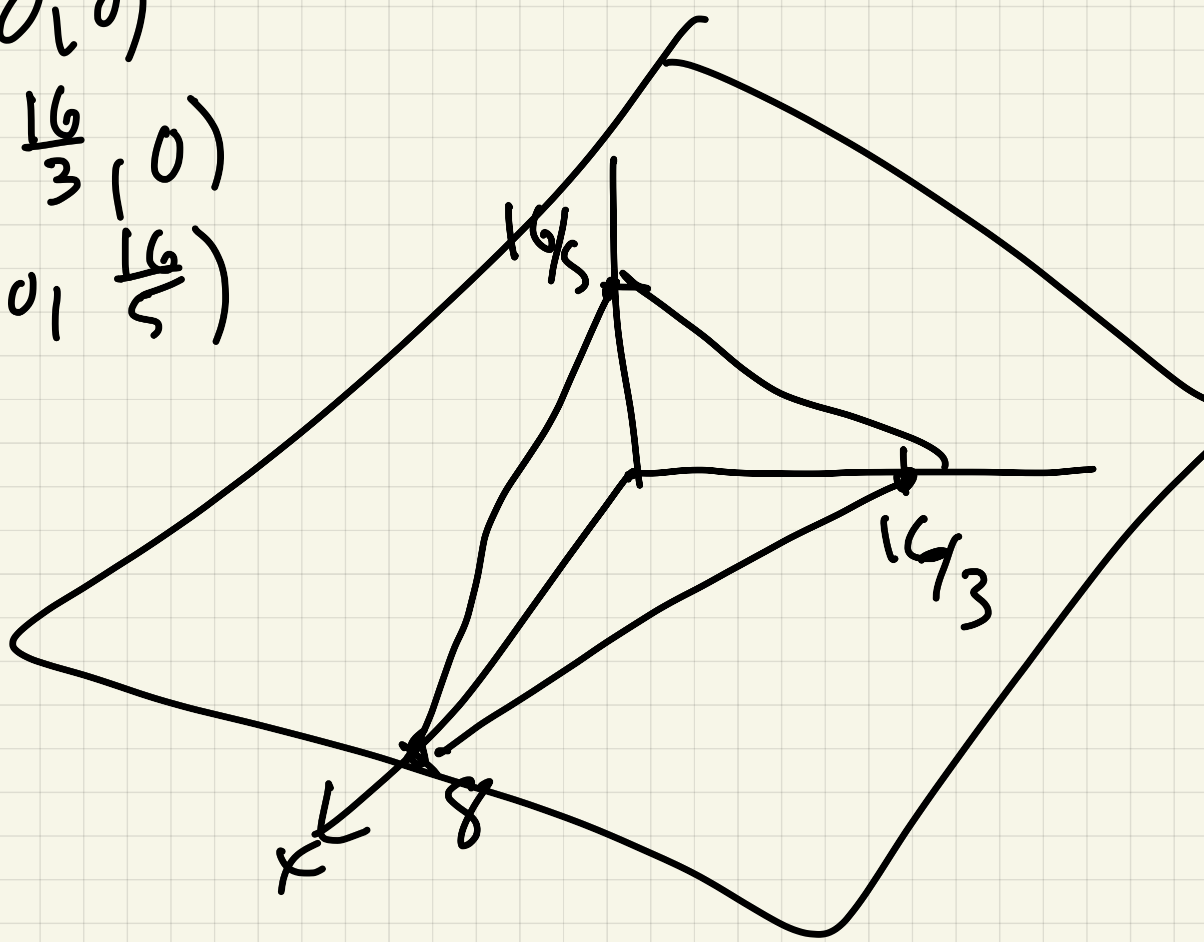
x- / y / z intercepts

$$2x+3y+5z=16$$

$(8,0,0)$

$(0,\frac{16}{3},0)$

$(0,0,\frac{16}{5})$



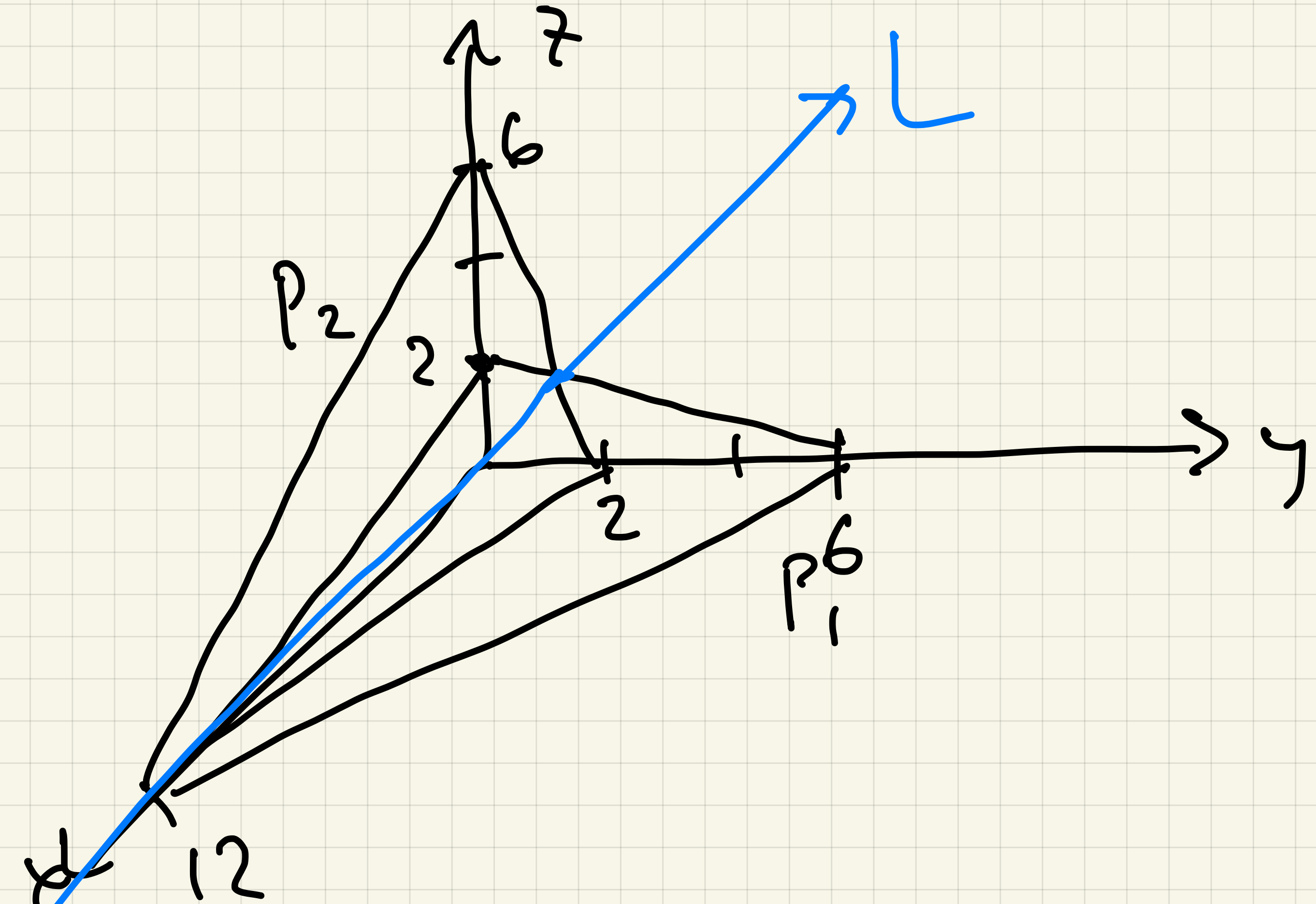
Ex Consider planes

$\bar{n}_1 = \langle 1,2,6\rangle$

$P_1: \quad x+2y+6z=12$

$P_2: \quad x+6y+2z=12$

$\bar{n}_2 = \langle 1,6,2\rangle$



(a) ~~P_1~~ Find the line L of intersection of P_1 & P_2

The direction of L is $\perp$ to $\bar{n}_1$ and to $\bar{n}_2$, so

$$L \parallel \bar{n}_1 \times \bar{n}_2$$

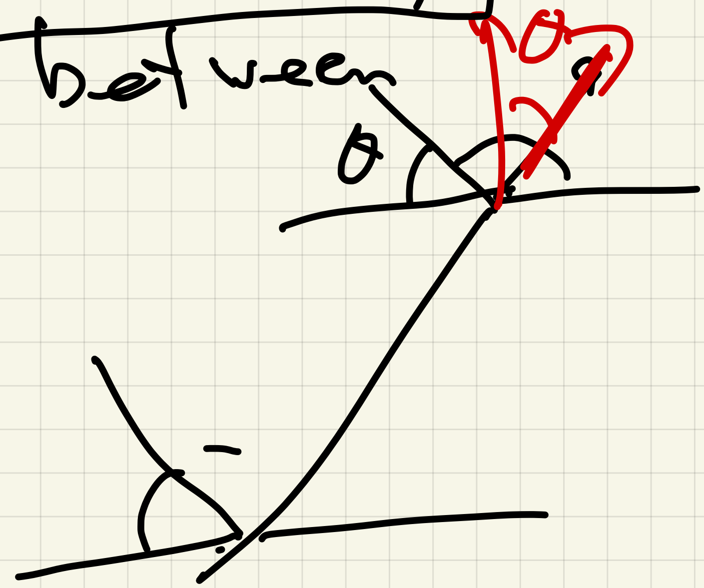
$$\bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 2 & 6 \\ 1 & 6 & 2 \end{vmatrix} = \langle -32, 4, 4 \rangle$$

direction of L

So $\bar{v} = \langle -32, 4, 4 \rangle \sim \langle -8, 1, 1 \rangle$

So $L : \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 - 8t \\ 0 + t \\ 0 + t \end{pmatrix}}$

(b) Find angle between P_1 & P_2



Use normal vector;

angle θ between P_1 & P_2 is

smallest between normal vector

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|} = \frac{25}{41}$$

(c) Where does L intersect

plane P_3 : $2x+y+8z=32$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-8t \\ t \\ t \end{pmatrix}$$

satisfies equation?

$$2(12-8t)+t+8t=32$$

$$24-16t+t+8t=32$$

$$-8=7t \qquad t=-8/7$$

So $P = \begin{pmatrix} 12-8(-8/7) \\ -8/7 \\ -8/7 \end{pmatrix} = \begin{pmatrix} \frac{148}{7} \\ -8/7 \\ -8/7 \end{pmatrix}.$