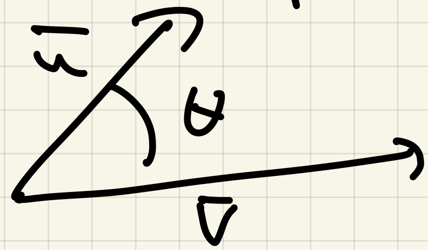
1/23/Calc3 Dot product

Last time

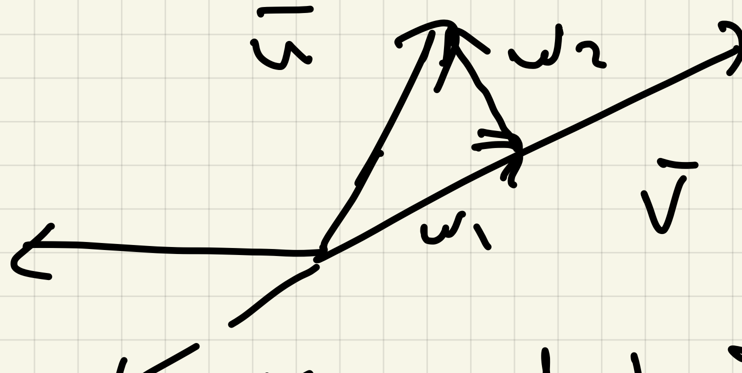


$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|}$$

$$\text{Proj}_{\bar{v}}\bar{u} = \frac{\bar{u}\cdot\bar{v}}{|v|^2}\bar{v}$$

$\shortparallel$

w_1



$\bar{w}_2 = \bar{u} - \bar{w}_1$

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component of $\bar{u}$ orthogonal to $\bar{v}$,

11.4 Defn

If $\bar{u} = \langle u_1, u_2, u_3\rangle$ & $\bar{v} = \langle v_1, v_2, v_3\rangle$ are vectors in $\mathbb{R}^3$, then the cross product of u and v is

$\bar{u} \times \bar{v} =$

$$\langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle$$

Tough to remember!

Easier to remember with <u>determinants</u>

<u>2×2 determinant</u> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

<u>Ex1</u> $\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 2 \cdot 7 - 3 \cdot 1 = 11$

$$\begin{vmatrix} 1 & 6 \\ 2 & 9 \end{vmatrix} = -3$$

<u>3×3 determints</u>

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

<u>Ex2</u>

(a) $\begin{vmatrix} 2 & 1 & 3 \\ 7 & 1 & 4 \\ 2 & 6 & 5 \end{vmatrix} =$

$$2\underbrace{\begin{vmatrix} 1 & 4 \\ 6 & 5 \end{vmatrix}} - 1\begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} 7 & 1 \\ 2 & 6 \end{vmatrix}$$

$$2(-19) - 1(27) + 3(40)$$

$$-38 - 27 + 120$$

$$-65 + 120 = 55$$

(b) $\begin{vmatrix} 2 & 3 & 7 \\ 4 & -1 & 6 \\ 2 & 0 & 5 \end{vmatrix} =$

$$2\begin{vmatrix} -1 & 6 \\ 0 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix} + 7\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= -10 - 24 + 14$$

$$= -34 + 14 = -20$$

Now the cross product is easy:

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ V_1 & v_2 & v_3 \end{vmatrix} =$$

$$i(u_2 v_3 - u_3 v_2) - j \ldots \text{etc.}$$

Ex 3 $\bar{u} = \langle 1, 2, 3 \rangle$
$\bar{v} = \langle 1, 0, 2 \rangle$

(a) $$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$i(4) - j(-1) + k(-2)$$

$$= \langle 4, 1, -2 \rangle$$

(b) $$\bar{v} \times \bar{u} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} =$$

$$i(-4) - j(1) + k(2) = \langle -4, -1, 2 \rangle$$

(c) $$\bar{v} \times \bar{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$i0 - j0 + k = \langle 0, 0, 0 \rangle$$

(d) $$i \times j = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$i0 - j0 + k1 = k$$

Algebraic properties

1. $\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})$
2. $\bar{u} \times \bar{0} = \bar{0}$
3. $\bar{u} \times \bar{u} = \bar{0}$

④ $c(\bar{u} \times \bar{v}) =$ $\quad c \in \mathbb{R}$

$$(c\bar{u}) \times \bar{v} = \bar{u} \times (c\bar{v})$$

distr. ⑤ $\bar{u} \times (\bar{v} + \bar{w}) =$

$$(\bar{u} \times \bar{v}) + (\bar{u} \times \bar{w})$$

⑥ $\bar{u} \cdot (\bar{v} \times \bar{w}) = (\bar{u} \times \bar{v}) \cdot w$

Geometric properties

Ⓐ $\bar{u} \times \bar{v} = 0 \iff \bar{u} \parallel \bar{v}$

$\left(\bar{u} = c\bar{v}, \ c \neq 0 \right.$
$\left. c \text{ number} \right)$

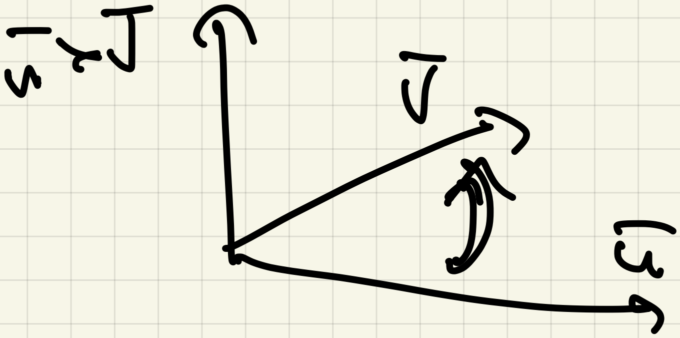
Ⓑ $\bar{u} \times \bar{v}$ is $\perp$ to $\bar{u}$ and $\bar{v}$!

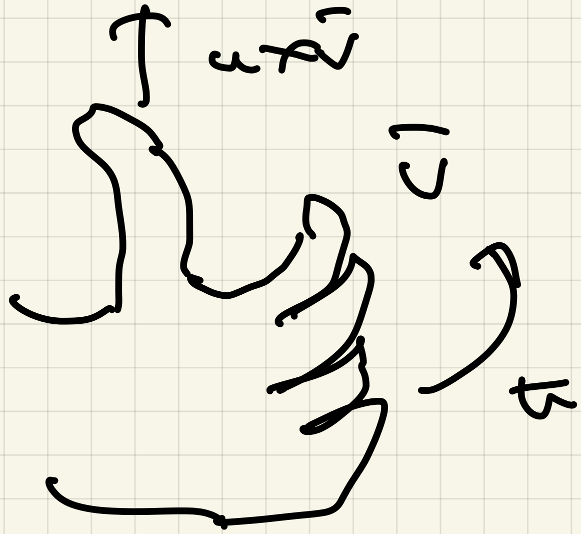
<u>why?</u> $\bar{u} \cdot (\bar{u} \times \bar{v}) \overset{⑥}{=} (\bar{u} \times \bar{u}) \cdot \bar{v} =$

$$\bar{0} \cdot \bar{v} = 0$$

<u>Moreover</u>, $\bar{u} \times \bar{v} \perp \bar{u}, \bar{v}$

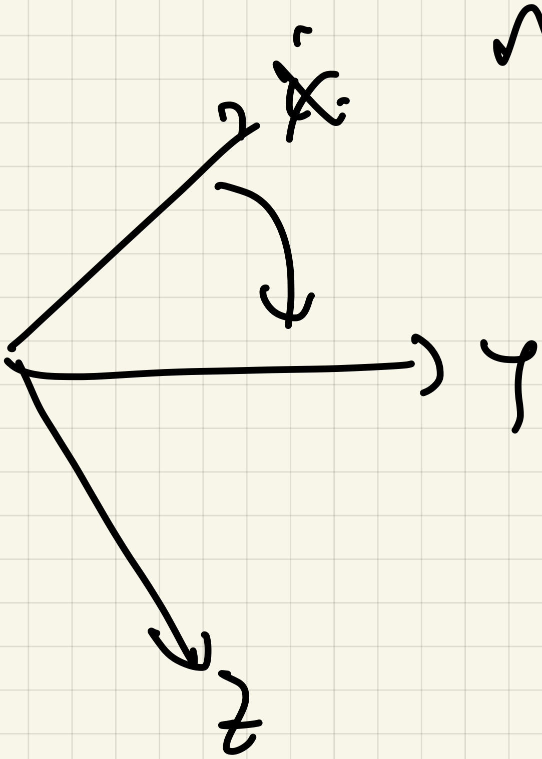
$\bar{u}, \bar{v}, \overline{u \times v}$ obey right hand rule



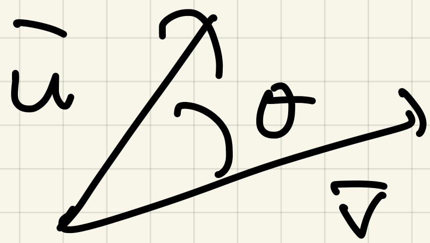


$\hat{i}\times\hat{j}=\hat{k}$

right hand



$\theta<\pi/2$



(c) $\|\bar{u}\times\bar{v}\| = \|\bar{u}\|\,\|\bar{v}\|\sin\theta$

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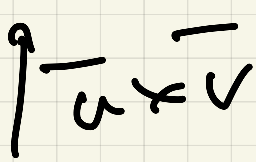
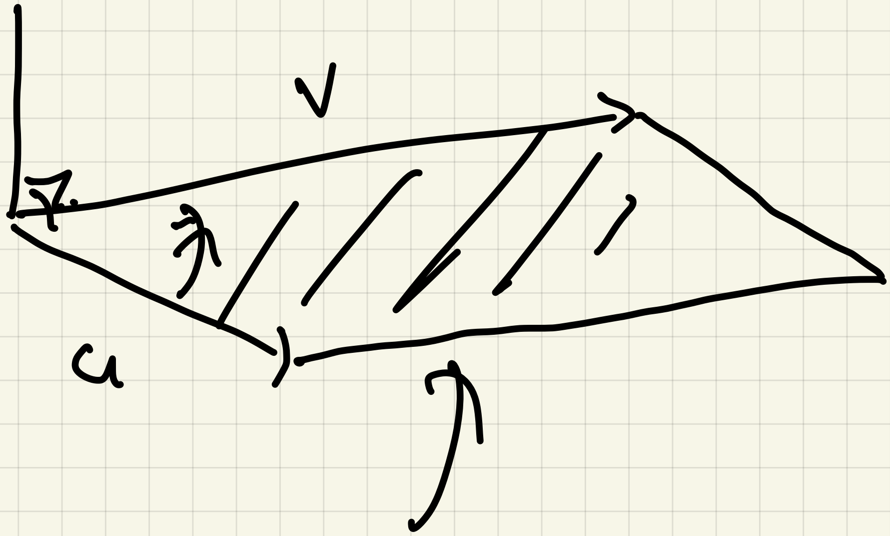
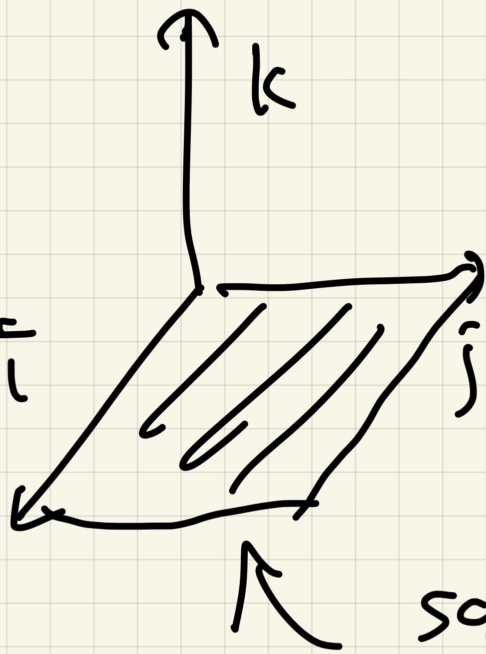
Area of the parallelogram
with edges $\bar{u}$ and $\bar{v}$





Area = $\|\vec{u} \times \vec{v}\|$

Ex 4

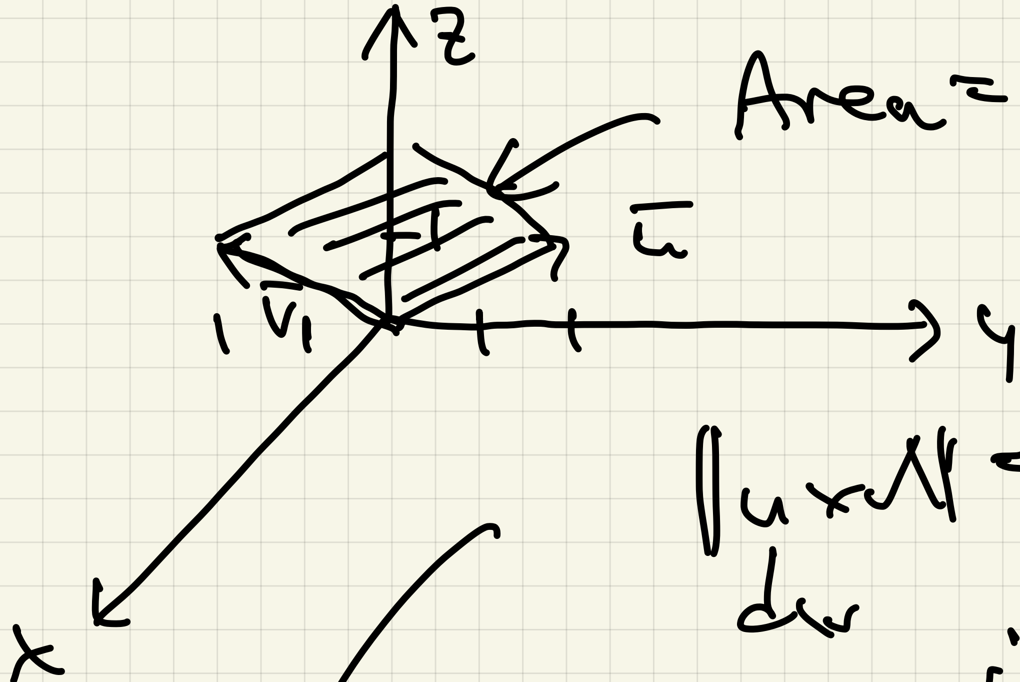


square of side 1

Area = 1

$\|i \times j\| = \|k\| = 1$

Ex 5



Area = 4

Geometry

$\|u \times v\| = 4$

dir $+x$ dir

$\therefore$

Ans

$\bar{u} \times \bar{v} = \langle 4, 0, 0 \rangle$

Algebra:

$\bar{u} = \langle 0, 2, 1 \rangle$

$\bar{v} = \langle 0, -2, 1 \rangle$

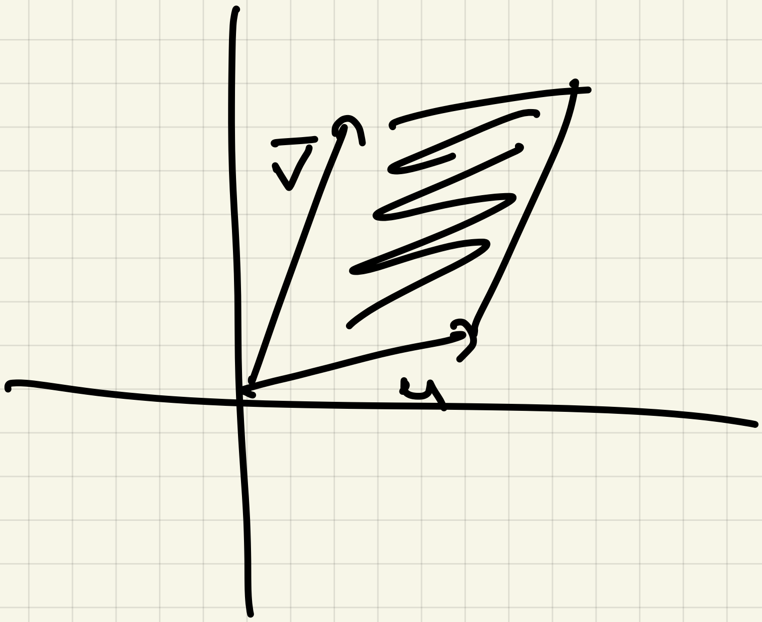
$$\bar{u} \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{vmatrix} =$$

$$i4 + 0j + 0k = \langle 4, 0, 0 \rangle$$

In general,

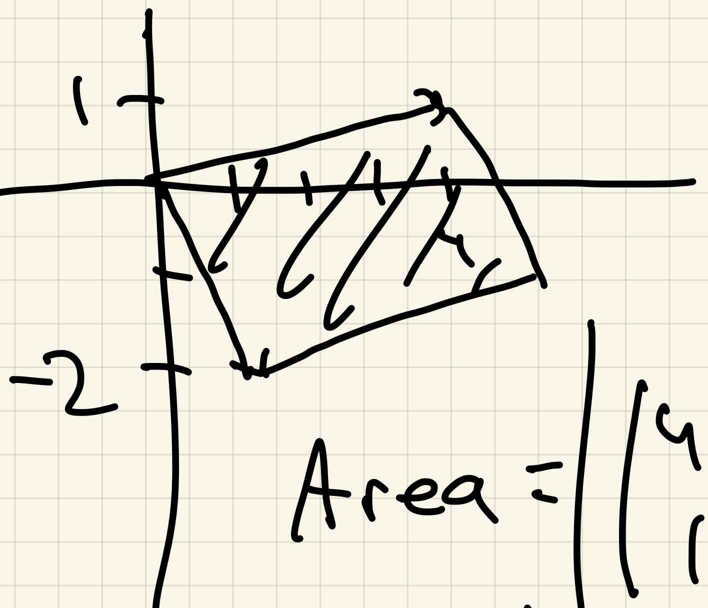
if $\bar{u} = \langle u_1, u_2 \rangle$

$\bar{v} = \langle v_1, v_2 \rangle$ in plane



Then $\left| \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right| =$ Area of parallelogram in $\mathbb{R}$

Ex 6



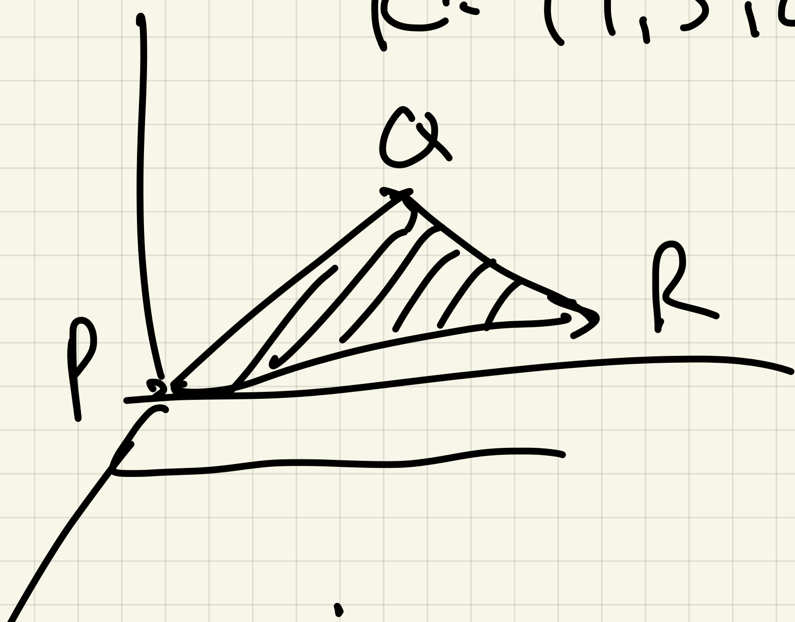
$u = \langle 4, 1 \rangle$

$v = \langle 1, -2 \rangle$

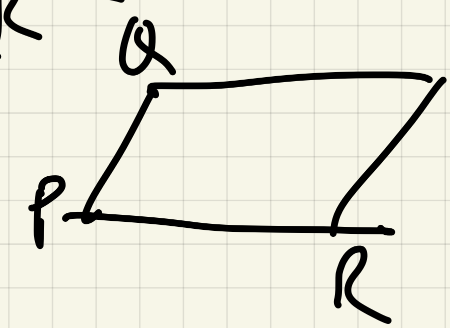
$$\text{Area} = \left| \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} \right| = |-8-1| = |-9| = 9$$

Ex7 Find area of the triangle with vertices

$P = (0,0,0)$, $Q = (-2,2,3)$, $R = (1,5,2)$



Area $\Delta PQR =$

$\frac{1}{2}$ Area 

$= \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| =$

$\frac{1}{2} \| \langle -2, 2, 3 \rangle \times \langle 1, 5, 2 \rangle \| =$

$\frac{1}{2} \sqrt{(-11)^2 + 7^2 + (-12)^2} =$

$$\begin{vmatrix} i & j & k \\ -2 & 2 & 3 \\ 1 & 5 & 2 \end{vmatrix} = \langle -11, 7, -12 \rangle$$

$$\frac{1}{2}\sqrt{121 + 49 + 144} =$$

$$\frac{1}{2}\sqrt{314}$$

<u>Defn</u>: <u>Triple scalar product</u>

if $\bar{u}, \bar{v}$ and $\bar{w}$ is

$$(\bar{u} \times \bar{v}) \cdot \bar{w}$$

<u>Note</u>: for easy calculation:

$$(\bar{u} \times \bar{v}) \cdot \bar{w} = \bar{u} \cdot (\bar{v} \times \bar{w}) =$$

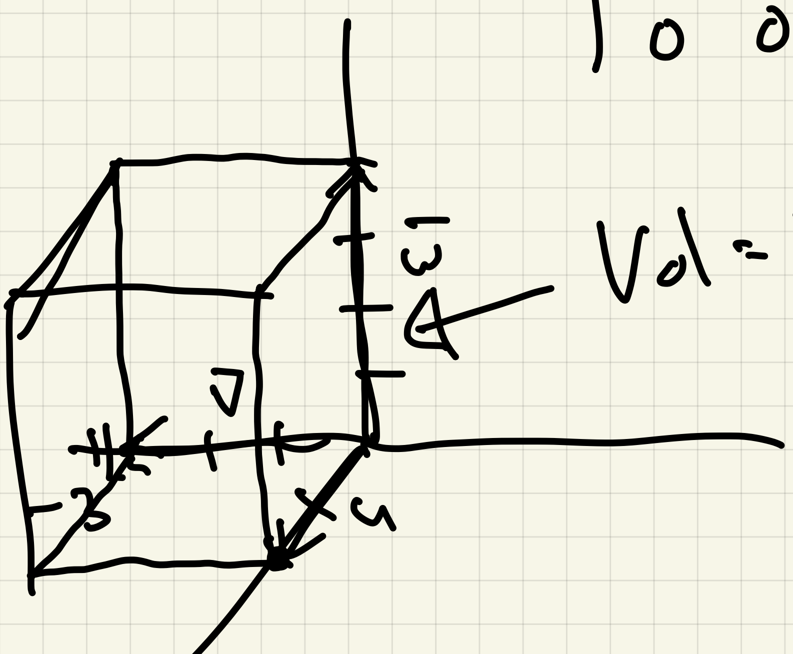
$$\langle u_1\, u_2\, u_3 \rangle \cdot$$

Calculate $\;=\;$ $\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Ex) $\bar{u} = \langle 2, 0, 0 \rangle$
$\bar{v} = \langle 0, -3, 0 \rangle$
$\bar{w} = \langle 0, 0, 4 \rangle$

$$(\bar{u} \times \bar{v}) \cdot \bar{w} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = \underline{\underline{-24}}$$



Vol = 24

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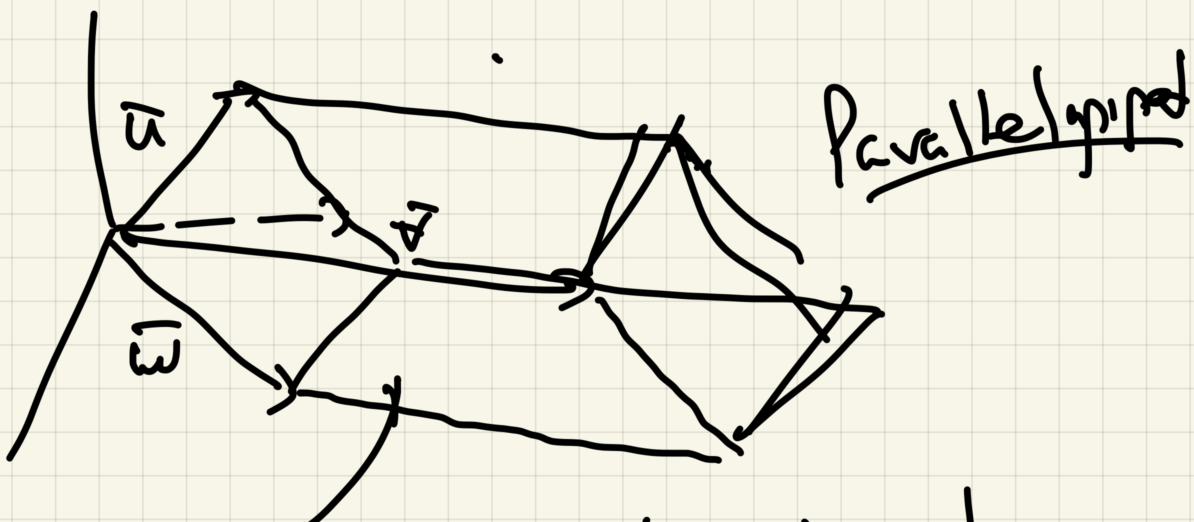
Geometrically

$|(\bar{u} \times \bar{v}) \cdot \bar{w}|$ = volume of "parallelapiped" spanned by $\bar{u}, \bar{v}, \bar{w}$,

Usually $\bar{u}, \bar{v}, \bar{w}$ not $\perp$, (in Ex))



Volume $= |(\bar{u} \times \bar{v}) \cdot \bar{w}|$