2/27/ Calc3

Quiz 9

4. $f(x,y) = \dfrac{x+2y^2}{5x+y}$

1. $\displaystyle \lim_{(x,y)\to(2,3)} \frac{x+2y^2}{5x+y} = \frac{20}{13}$

2. Continuous:

$\{(x,y) : 5x+y \neq 0\}$

3. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x+2y^2}{5x+y} =$

limit along x-axis $(y=0)$

$\displaystyle \underbrace{\lim_{x\to 0} \frac{x}{5x} = \lim_{x\to 0} \frac{1}{5}}_{\text{same for } x\neq 0} = \frac{1}{5}$

4. limit along y-axis

Set $x=0$

$$\lim_{y\to 0} \frac{2y^2}{y} = \lim_{y\to 0} \frac{2y}{1} = 0$$

5. $$\lim_{(x,y)\to(0,0)} \frac{\cancel{x}+2y^2}{5x+y}$$ exists?

~~yes~~ lim DNE b/c limits along different paths are $0 \neq \frac{1}{5}$

<u>Last time</u> Chain Rule
Implicit differentiation

<u>Ex)</u> $\cos xy + \sin yz + wz = 20$

<u>Defines $w$</u> as a function <u>of $x, y, z$</u> near $(1, 0, 19, 1)$

( <u>in fact,</u> $w = \dfrac{20 - \cos xy - \sin yz}{z}$ )

Find $\partial w/\partial x$, $\partial w/\partial y$, $\partial w/\partial z$ at

$$(1, 0, 19, 1)$$
$$x \quad y \quad z \quad w$$

Think of $w$ as ~~function~~ $f(x, y, z)$

$$\partial/\partial x : \cos xy + \underline{\sin yz} + wz = 20$$

$$\underline{-y \sin xy} + 0 + z\, \partial w/\partial x = 0$$

$$\partial w/\partial x = \frac{y \sin xy}{z}\Bigg|_{(1,0,19,1)} = \frac{0}{19} = 0$$

$$\partial/\partial y : -x \sin xy + z \cos yz + z\, \partial w/\partial y = 0$$

so $$\partial w/\partial y = \frac{x \sin xy - z \cos yz}{z}\Bigg|_{(1,0,19,1)}$$

$$= \frac{-19}{19} = -1$$

$$\partial/\partial z : 0 + y \cos yz + \underbrace{\partial w/\partial z \cdot z + w \cdot 1}_{\text{prod rule}} = 0$$

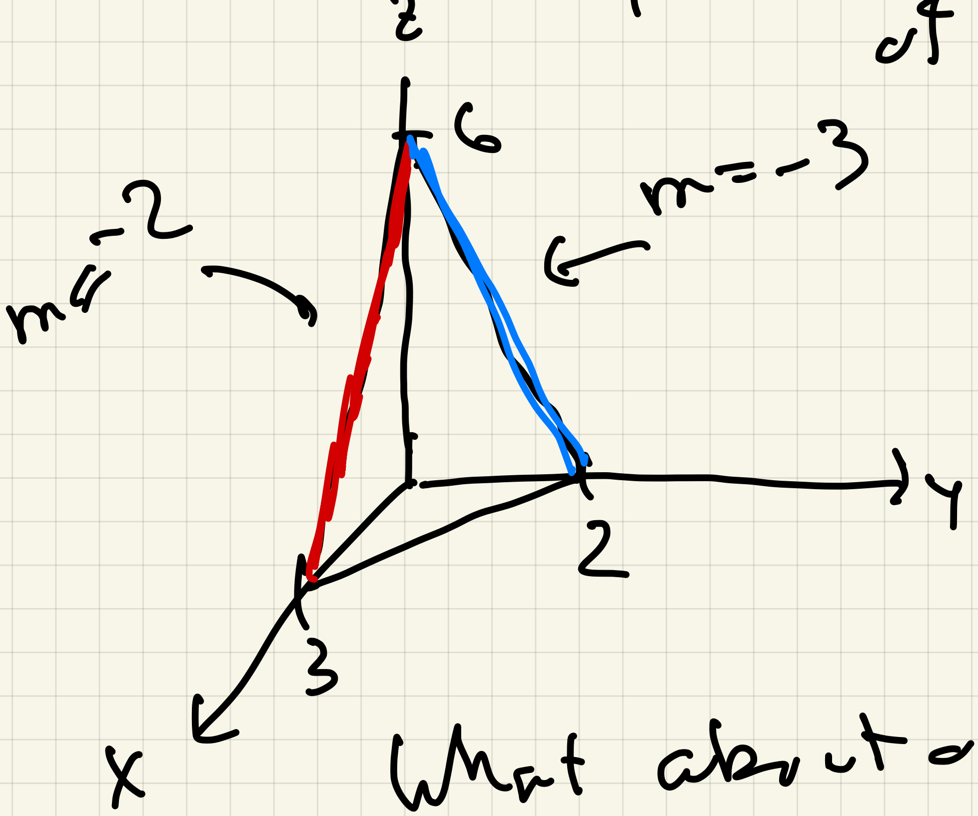
So $\partial w/\partial z = \dfrac{-y\cos yz - w}{z}\Big|_{(1,0,9,1)} = \dfrac{-1}{19}$

## §13.5

If $z = f(x,y)$

$\partial f/\partial x\,(a,b) =$ rate of change of $z$ <u>wrt</u> $x$

$\partial f/\partial y\,(a,b) =$ rate of change of $z$ w.r.t to $y$




$z = 6 - 2x - 3y$

$\partial z/\partial x = -2$

What about other <u>directions</u>?

For a unit vector $u = \langle u_1, u_2 \rangle$ notice that

$$\bar{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + u_1 t \\ b + u_2 t \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} a \\ b \end{pmatrix}} + t\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

parametrizes a line through $(a,b)$ in direction $\bar{u}$

at speed $= 1$

<u>Defn</u> : The <u>directional derivative of $z = f(x,y)$ at $(a,b)$ in direction $\bar{u}$</u> is

$$D_u f(a,b) = \frac{d}{dt}\Big( f(\underbrace{a + u_1 t}_{x}, \underbrace{b + u_2 t}_{y})\Big)\Big|_{t=0}$$

$$\parallel$$

$$\partial f/\partial x\,(a,b) \cdot \frac{dx}{dt} + \partial f/\partial y\,(a,b) \cdot \frac{dy}{dt}$$

$$\parallel$$

$$f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$

$$\parallel$$

$$\underbrace{\langle f_x(a,b), f_y(a,b)\rangle}_{\substack{\text{Gradient of } f \\ \nabla f(a,b)}} \cdot \underbrace{\langle u_1, u_2\rangle}_{\bar{u}}$$

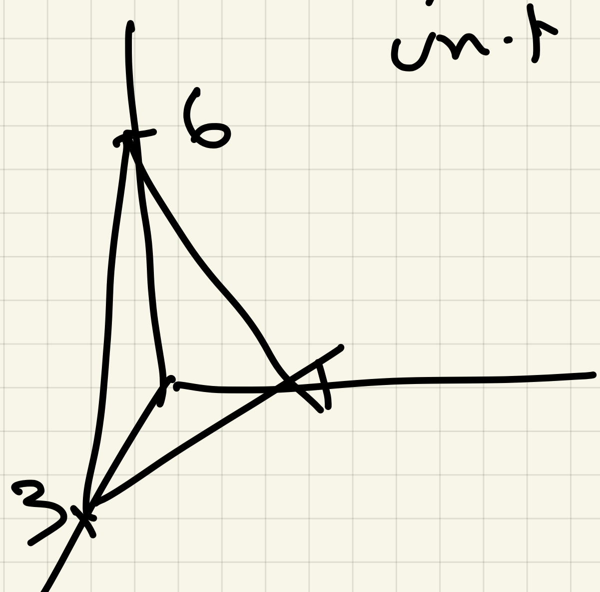
<u>Definition</u>: The <u>gradient</u> of $f$ at $(a,b)$ is

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

Formula for directional derivative

$$D_{\bar{u}} f(a,b) = \nabla f(a,b)\cdot \underbrace{\bar{u}}_{\text{unit vector}}$$

<u>Ex①</u>:



$$f(x,y) = 6 - 2x - 3y$$

$$\nabla f = \langle -2, -3 \rangle$$

Find directional derivatives in directions:

| $(1,0)$ | $(0,1)$ | $(-1,0)$ | $(2,1)$ | $(-3,2)$ | $(2,3)$ |
|---|---|---|---|---|---|
| $-2$ | $-3$ | $2$ | $-7/\sqrt{5}$ | $0$ | $-\sqrt{13}$ |

$\langle -2,-3 \rangle \cdot \langle 1,0 \rangle$

$\langle -2,-3 \rangle \cdot \dfrac{\langle 2,1 \rangle}{\sqrt{5}}$

$\langle -2,-3 \rangle \cdot \langle 0,1 \rangle$

$\langle -2,-3 \rangle \cdot \dfrac{\langle -3,2 \rangle}{\sqrt{13}}$

<u>Ex!</u> $z = f(x,y) = \underline{25 - x^2 - y^2}$

(a) Find $\nabla f$ at $(3,4)$

(b) Find $D_u f(3,4)$ in direction

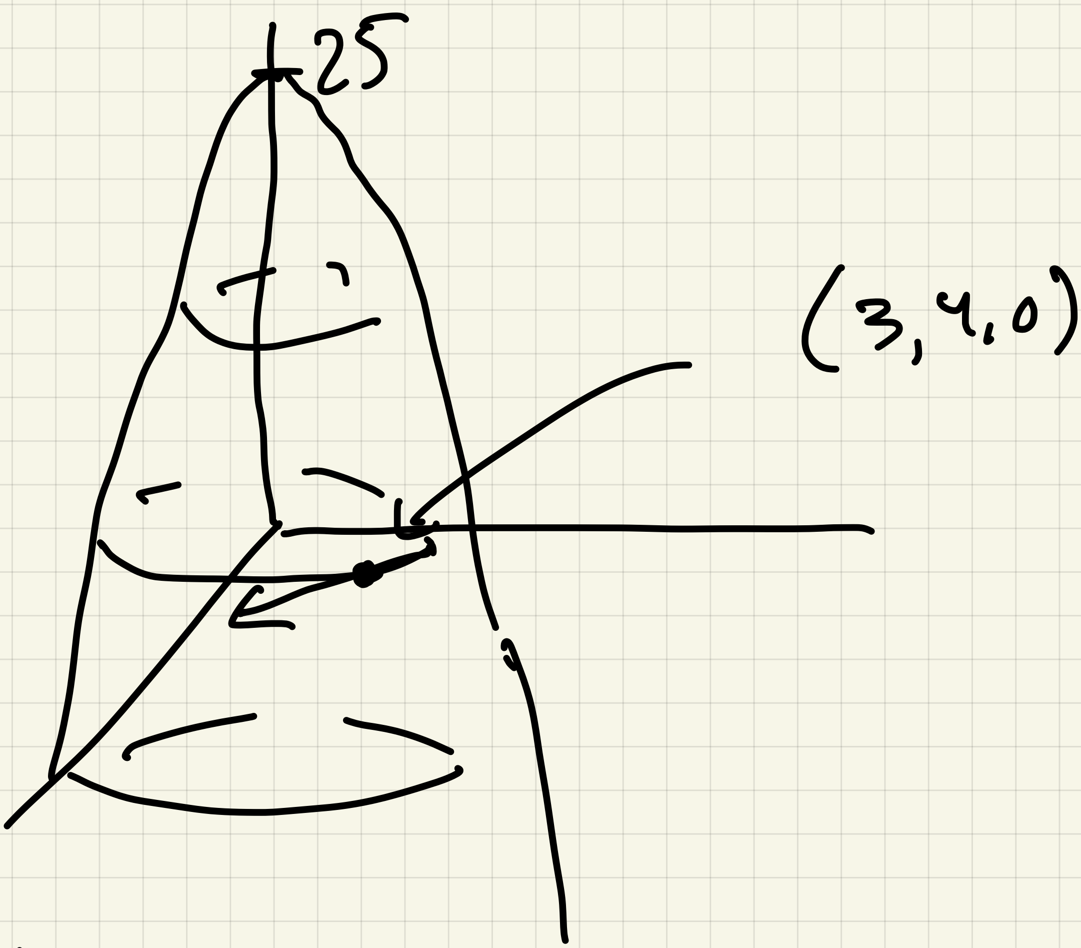
pos x axis

pos y axis

$\langle 1,1 \rangle \qquad \langle 4,-3 \rangle$

$\nabla f = \langle -2x, -2y \rangle$

$\nabla f(3,4) = \langle -6, -8 \rangle$




$D_u f(3,4)$ :

pos x : $u = \langle 1,0 \rangle$

so $D_u f(3,4) = -6$

pos y : $u = \langle 0,1 \rangle$

$D_u f(3,4) =$

$(-6,-8)\cdot(0,1) = -8$

$\langle 1,1 \rangle$ : $\langle -6,-8 \rangle \cdot \frac{\langle 1,1 \rangle}{\sqrt{2}} =$

$\frac{-14}{\sqrt{2}} = -9.899$

$\langle 4,-3 \rangle$ : $\langle -6,-8 \rangle \cdot \frac{\langle 4,-3 \rangle}{5} = 0$

<u>Notice</u>: If $\theta$ is angle between $\bar{u}$ and $\nabla f(a,b)$, then

$$\cos\theta = \frac{\nabla f(a,b)\cdot u}{|\nabla f(a,b)|\,|u|} \quad \nabla f(a,b)\cdot u = D_u f(a,b).$$

$|u| = 1$

$$\boxed{D_u f(a,b) = |\nabla f(a,b)| \cdot \cos\theta}$$

<u>Consequences</u>:

① Direction of maximal increase is $\bar{u} = \nabla f(a,b)$

② Direction of maximal decrease is $\bar{u} = -\nabla f(a,b)$

$$\boxed{③ \quad \bar{u} \perp \nabla f(a,b) \Rightarrow \cos\theta = 0 \qquad D_{u} f(a,b) = 0}$$

In Ex1 we saw ③ $\langle 4, -3 \rangle$

direction $\langle 1,1 \rangle$ $-\frac{14}{\sqrt{2}}$ $-9,89$

but maximal rate of decrease is in direction

$-\nabla f(a,b) = -\langle -6, -8 \rangle =$

$\langle 6, 8 \rangle$

actual rate of change is

$$\langle -6,-8\rangle \cdot \frac{\langle 6,8\rangle}{\sqrt{100}} = \frac{-36-84}{10} =$$

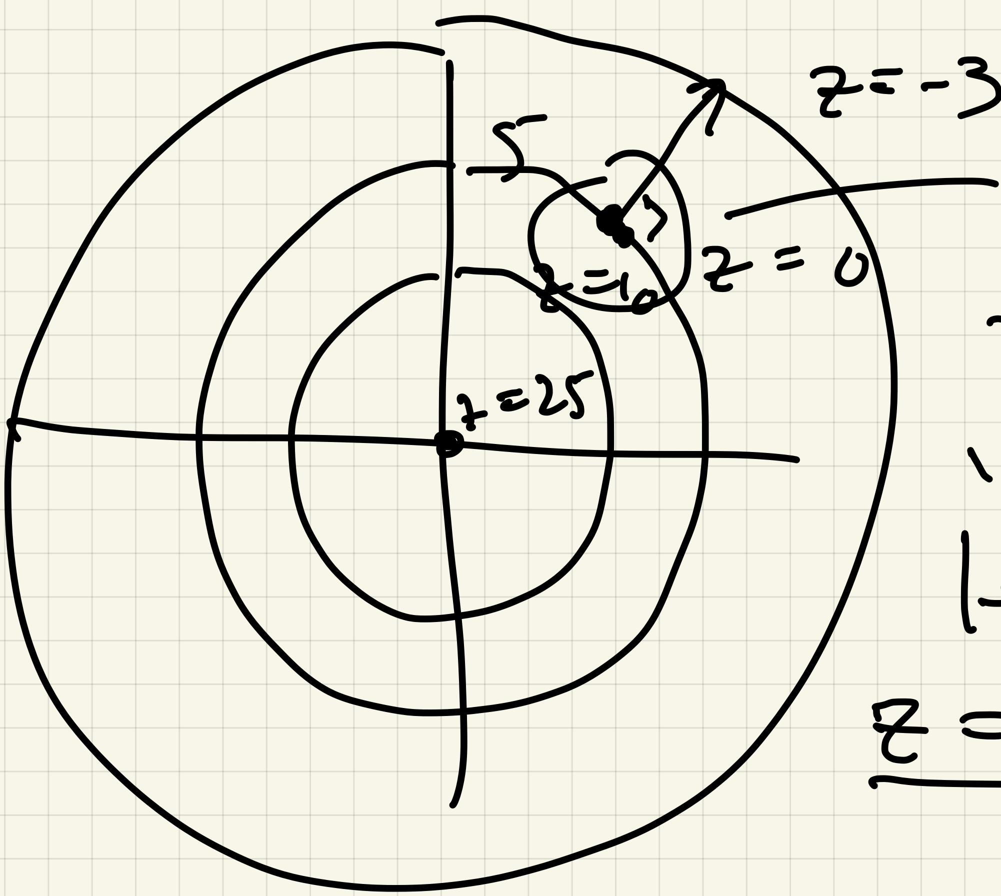
$$-\frac{100}{10} = -10$$

Similarly, direction increase

is $\langle -6,-8\rangle$

rate of change 10

Level curves for $z = 25-x^2-y^2$




In general

The gradient is $\perp$ to level sets

$z$ = constant

**Illustration:**

Ex $z = q(x,y) = y - x^2$

a) find $\nabla q$

b) Compute $\nabla q(a,b)$ at

$(a,b) = (0,0)$, $(1,1)$, $(2,4)$

$\left(\frac{1}{2}, \frac{1}{4}\right)$, $(-1,1)$

**Notice:** each points is on the parabola $y - x^2 = 0$

$\nabla q = \langle -2x, 1 \rangle$

$\boxed{\nabla q(0,0) = \langle 0,1 \rangle}$ $x =$

$\nabla q(1,1) = \langle -2,1 \rangle$

$\nabla g(2,4) = \langle -4, 1 \rangle$

$\nabla g\left(\frac{1}{2}, \frac{1}{4}\right) = \langle -1, 1 \rangle$

$\nabla g(-1,1) = \langle 2, 1 \rangle$