2/18 | Calc 3

13.1 $z = f(x, y)$

Domain.
Graph
Range

Ex1 Find domain & range

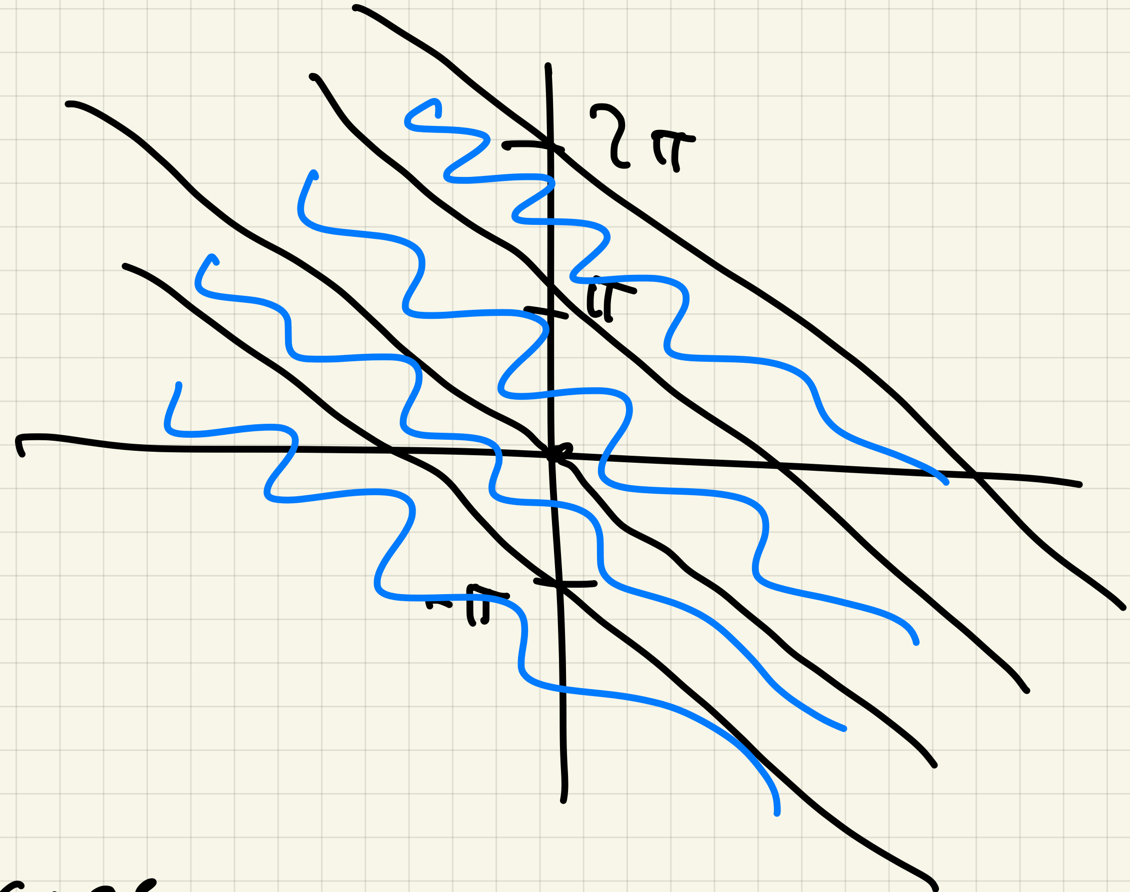
$$z = \frac{1}{\sin(x+y)} = \csc(x+y)$$

Domain: need $\sin(x+y) \neq 0$

i.e. $x+y \neq 0, \pm\pi, \pm 2\pi, \ldots$

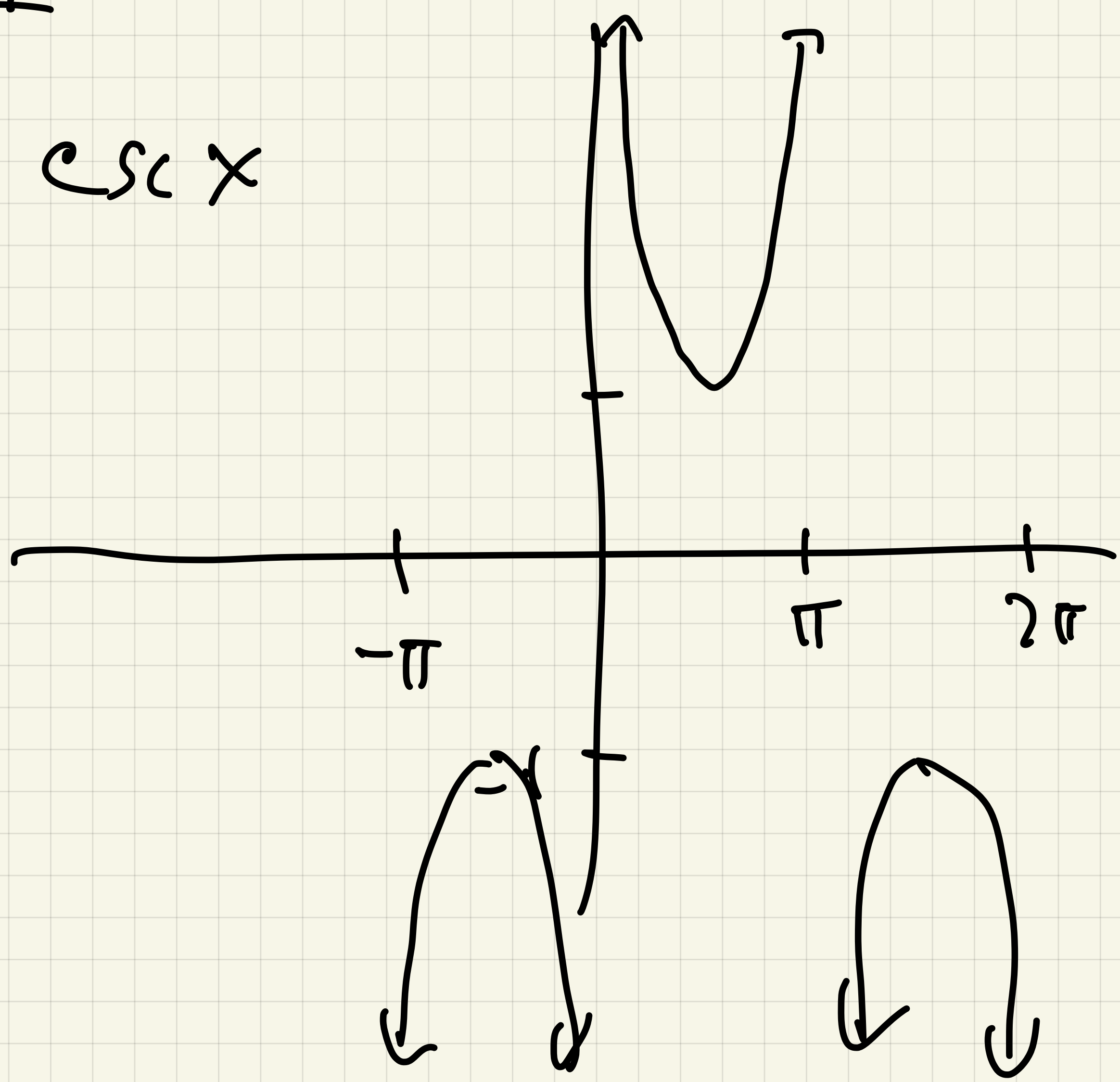
Domain: $\{(x,y) : \underline{x+y \neq n\pi}, n \in \mathbb{Z}\}$

Domain



Range

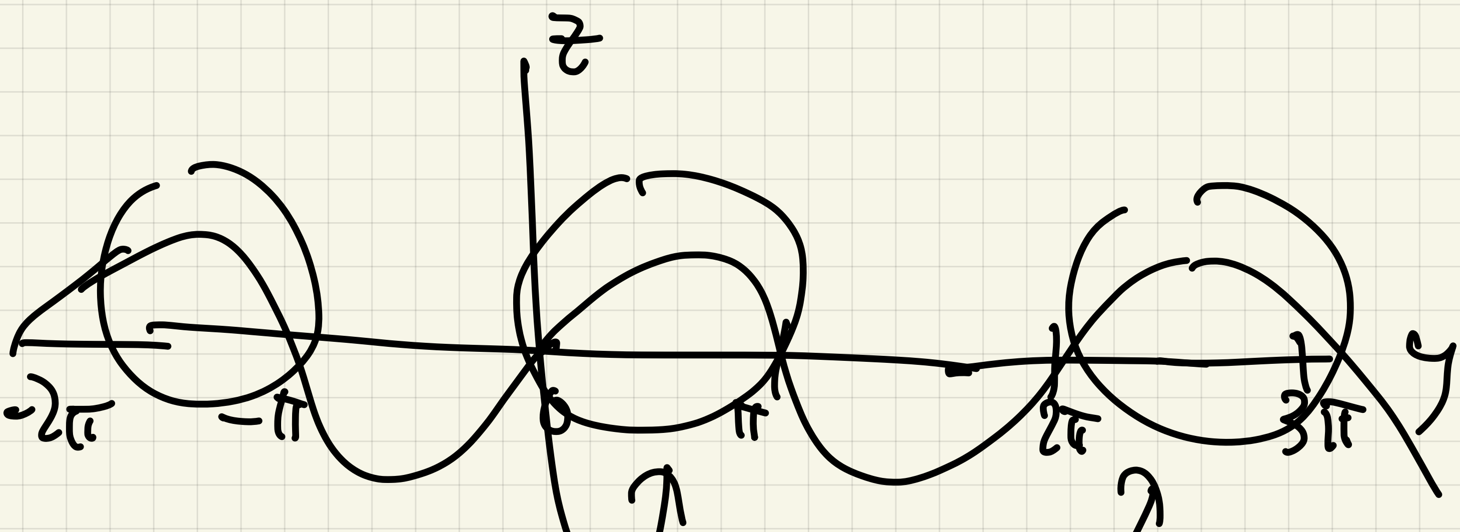
$y = \csc x$



So Range $(-\infty,-1]\cup[1,\infty)$

(b) $$z=\sqrt{\sin\left(\sqrt{x^2+y^2}\right)}$$

Need $\sin\left(\sqrt{x^2+y^2}\right)\geq 0$



So Domain: $\{(x,y):$

$0\leq\sqrt{x^2+y^2}\leq\pi$

$2\pi\leq\sqrt{x^2+y^2}\leq 3\pi$

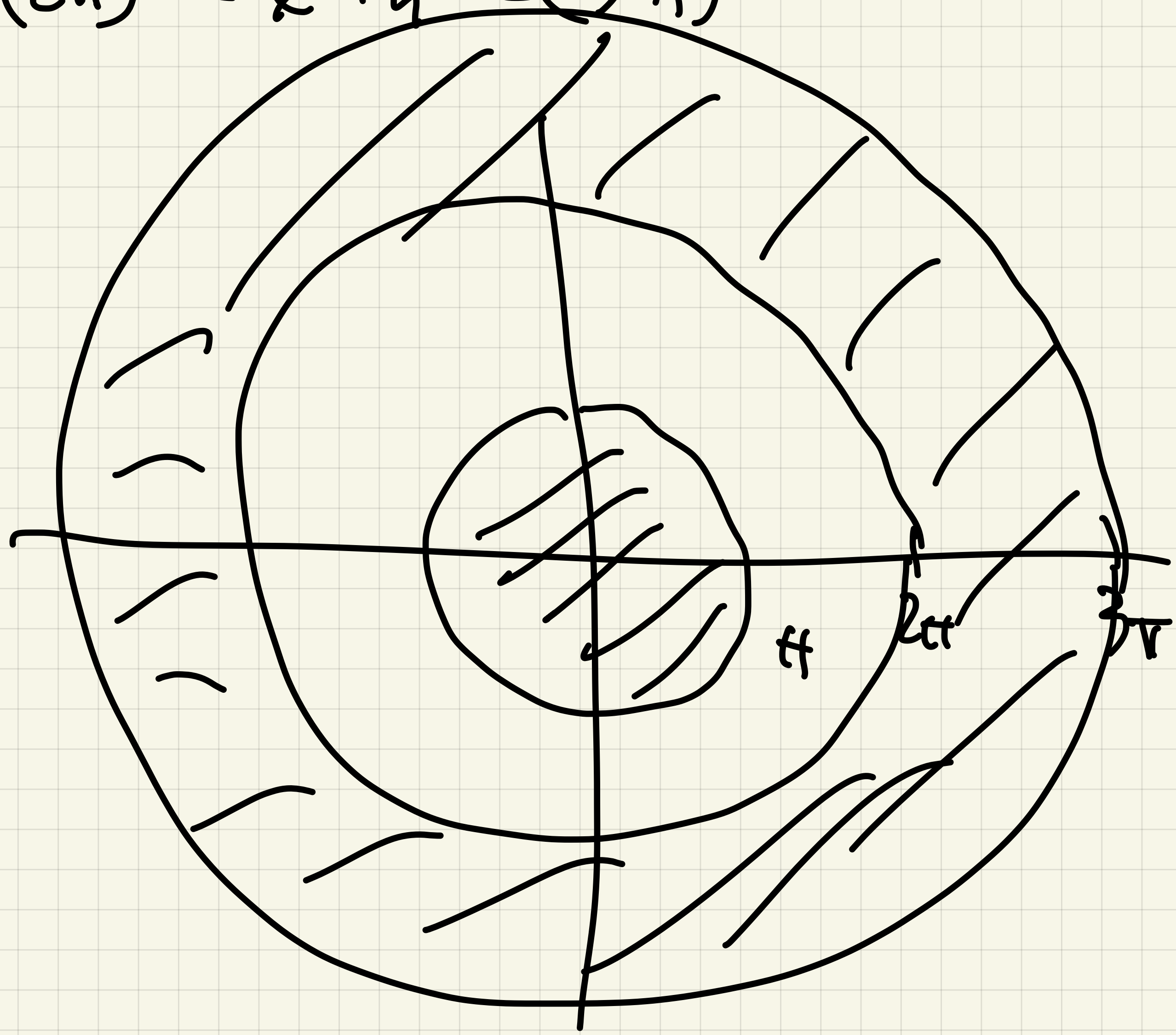
$\vdots$

$\}$

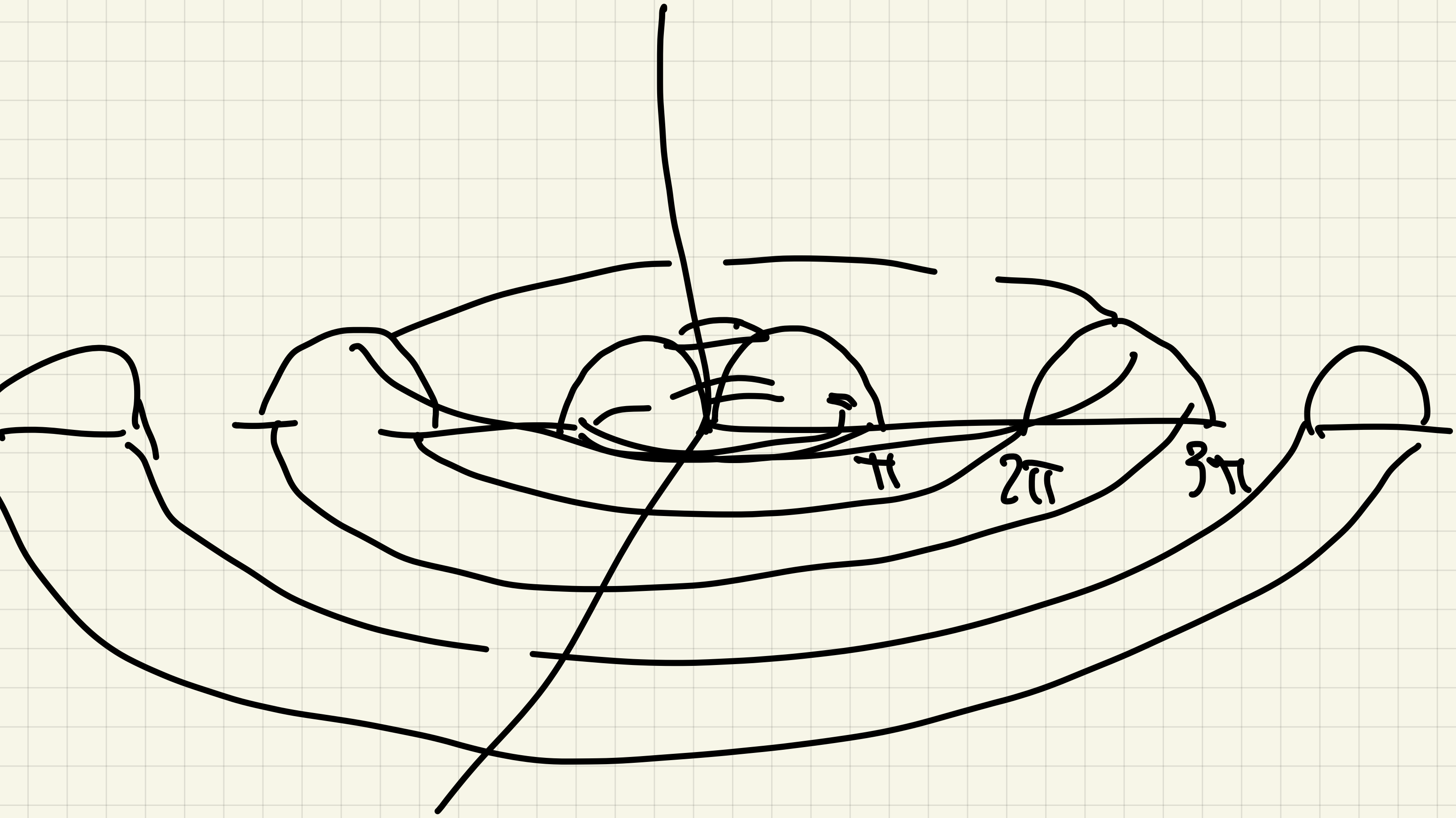
$$\left\{ (x,y) : 2n\pi \le \sqrt{x^2+y^2} \le (2n+1)\pi \;\middle|\; n \ge 0,\, n \in \mathbb{Z} \right\}$$

$x^2+y^2 \le \pi^2$

$(2\pi)^2 \le x^2+y^2 \le (3\pi)^2$



range $[0,1]$

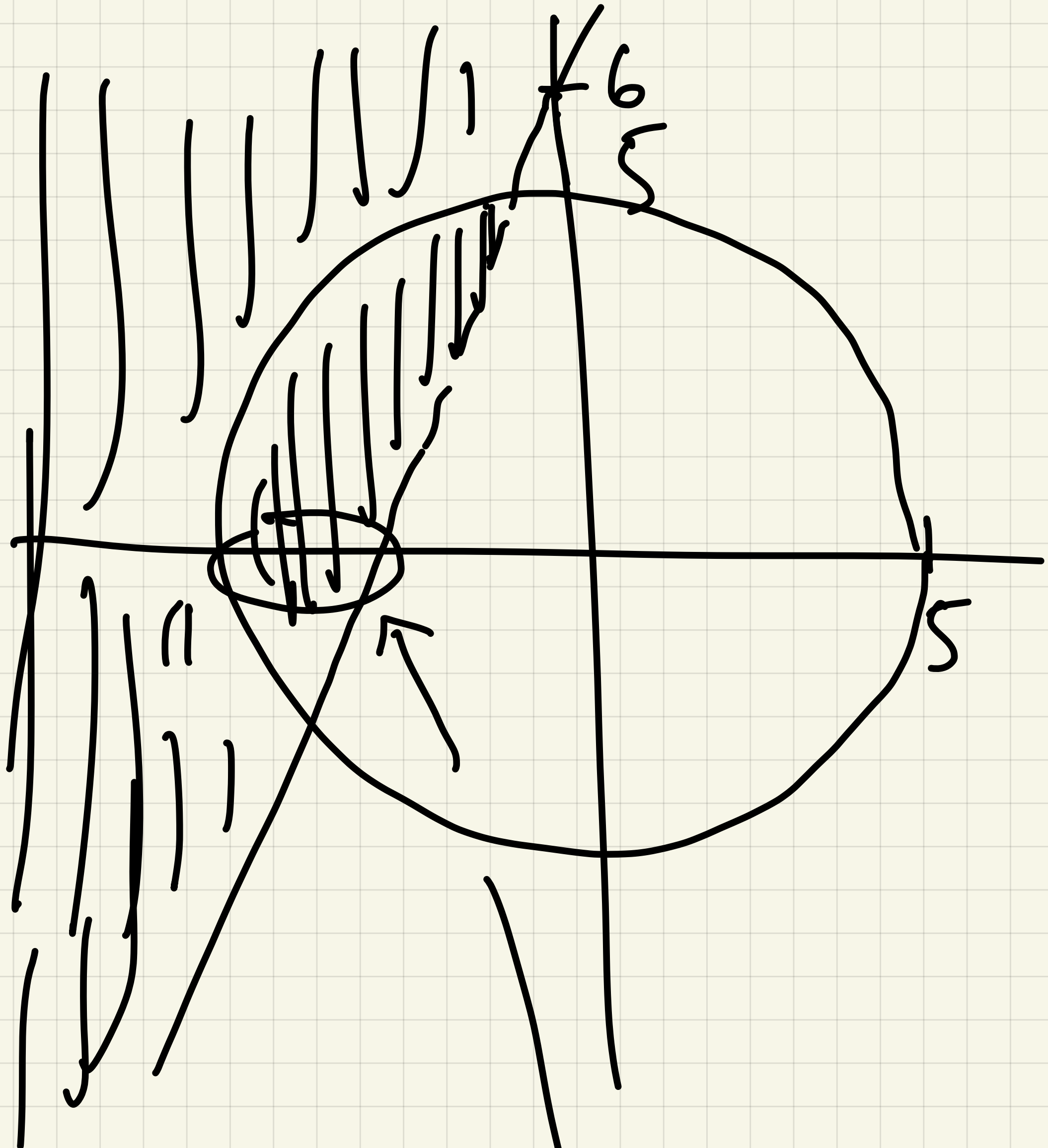


(c) $z = \dfrac{\sqrt{y-3x-6}}{25-x^2-y^2}$

Need: $y-3x-6 \geq 0$

$25-x^2-y^2 \neq 0$

$y \geq 3x+6, \quad x^2+y^2 \neq 25$



Range?

Inside circle $-5 < x < -2, \; y = 0$

$$z = \frac{\sqrt{-3x-6}}{25 - x^2}$$

$-5 < x < 0$

$$\lim_{x \to -5^+} \frac{\sqrt{-3x-6}}{25-x^2} = \frac{\sqrt{21}}{0^+} \to +\infty$$

$$z\big|_{x=-2} = 0$$

So range $[0, \infty)$

Outside ~~scircle~~ : $(-\infty, 0)$

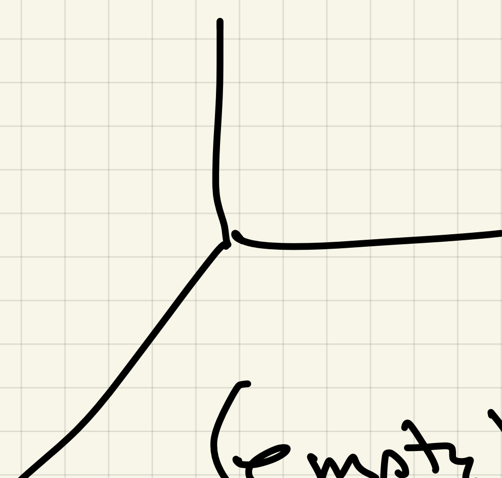
together:

range $(-\infty, \infty)$

Ex 2 Sketch level sets for

(a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

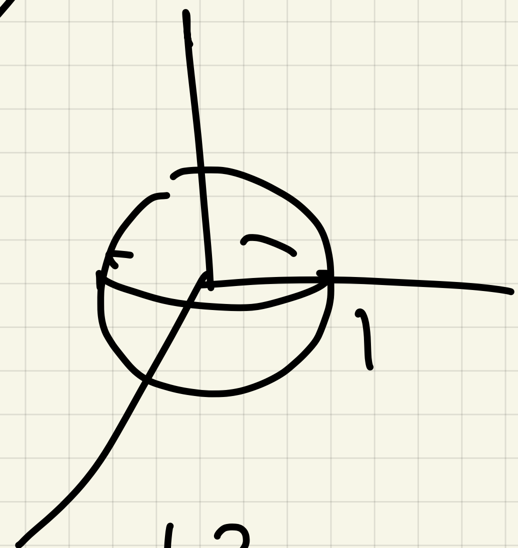
~~f=~~ $t = -1$



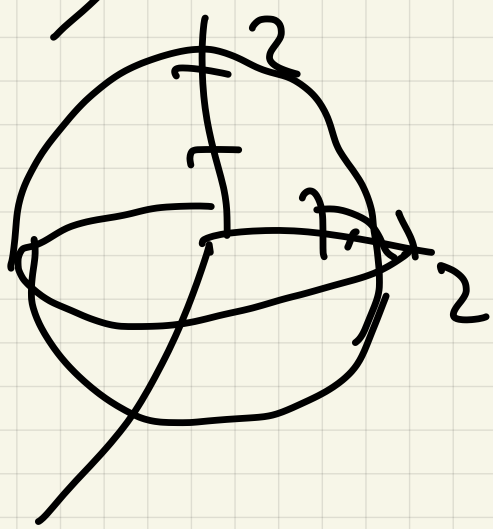
(empty)

$\underline{t = 0}$

$\underline{t = 1}$



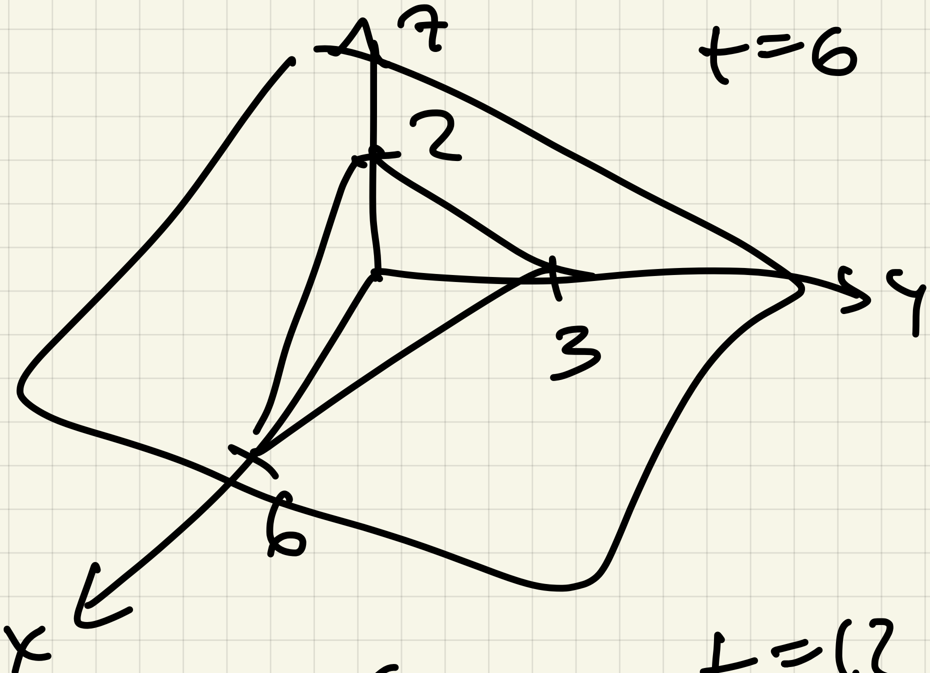
$\underline{t = 2}$



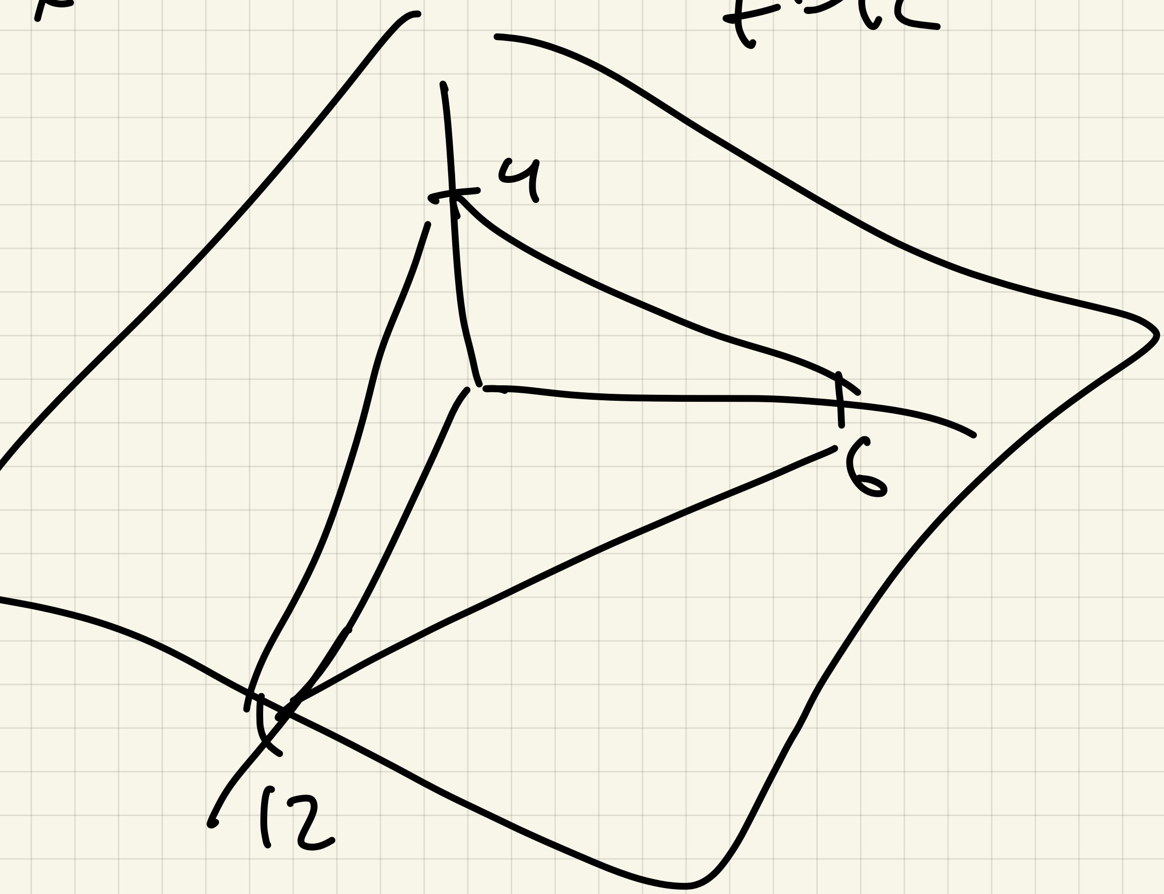
$\vdots$

(b) $t = x + 2y + 3z$

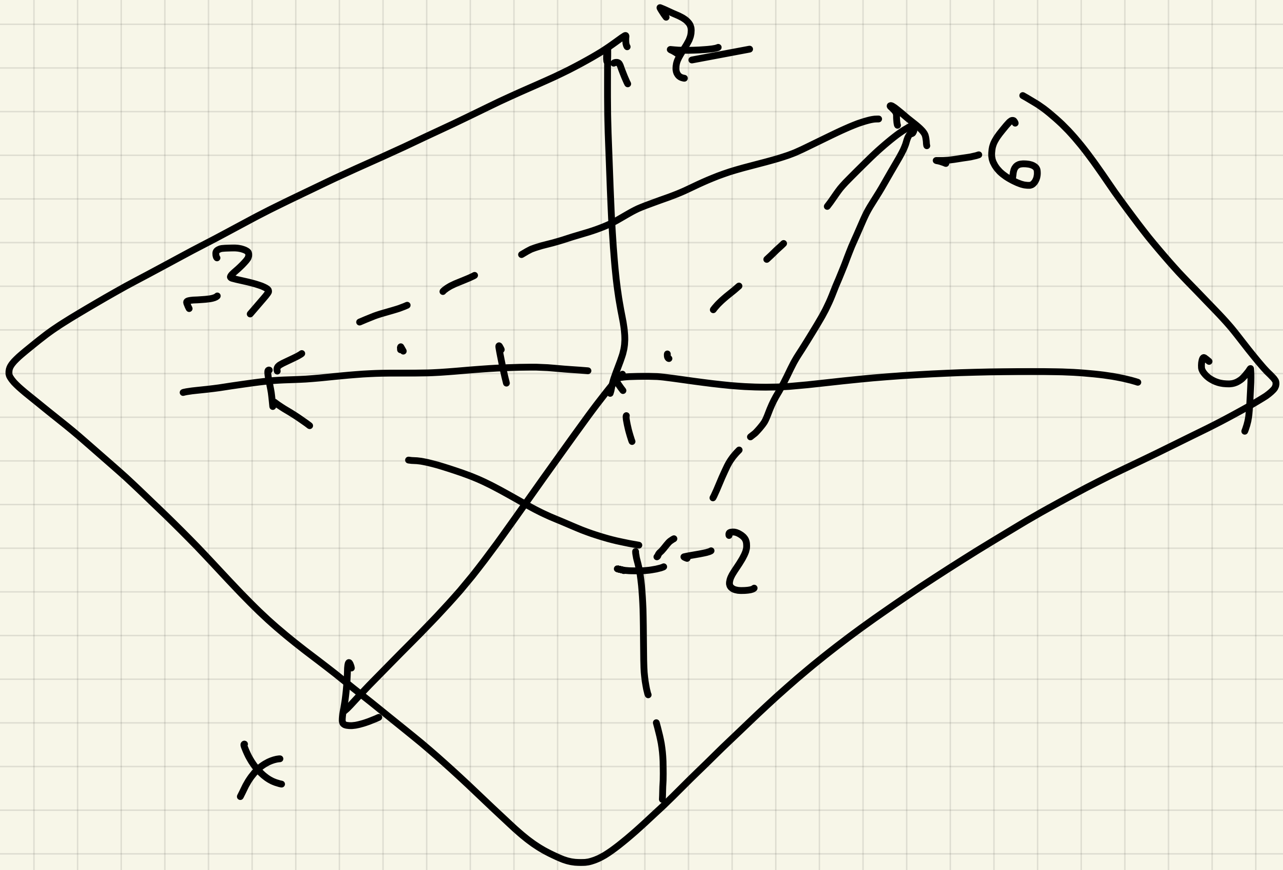
$t=6$



$t=12$



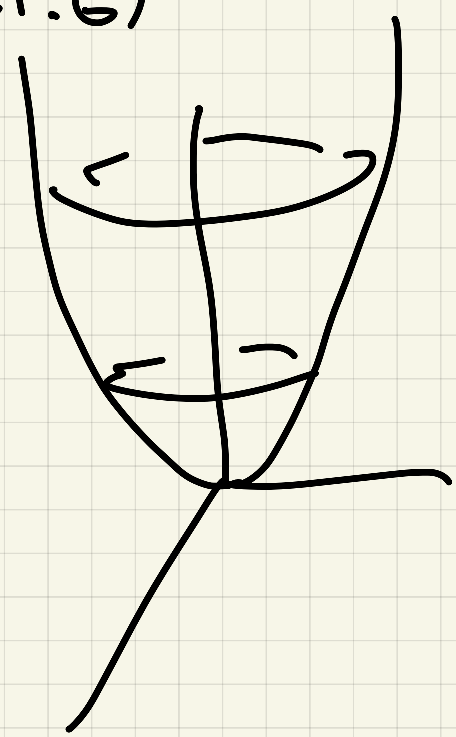
$t=-6$



parallel planes

Quadrics (§ 11.6)

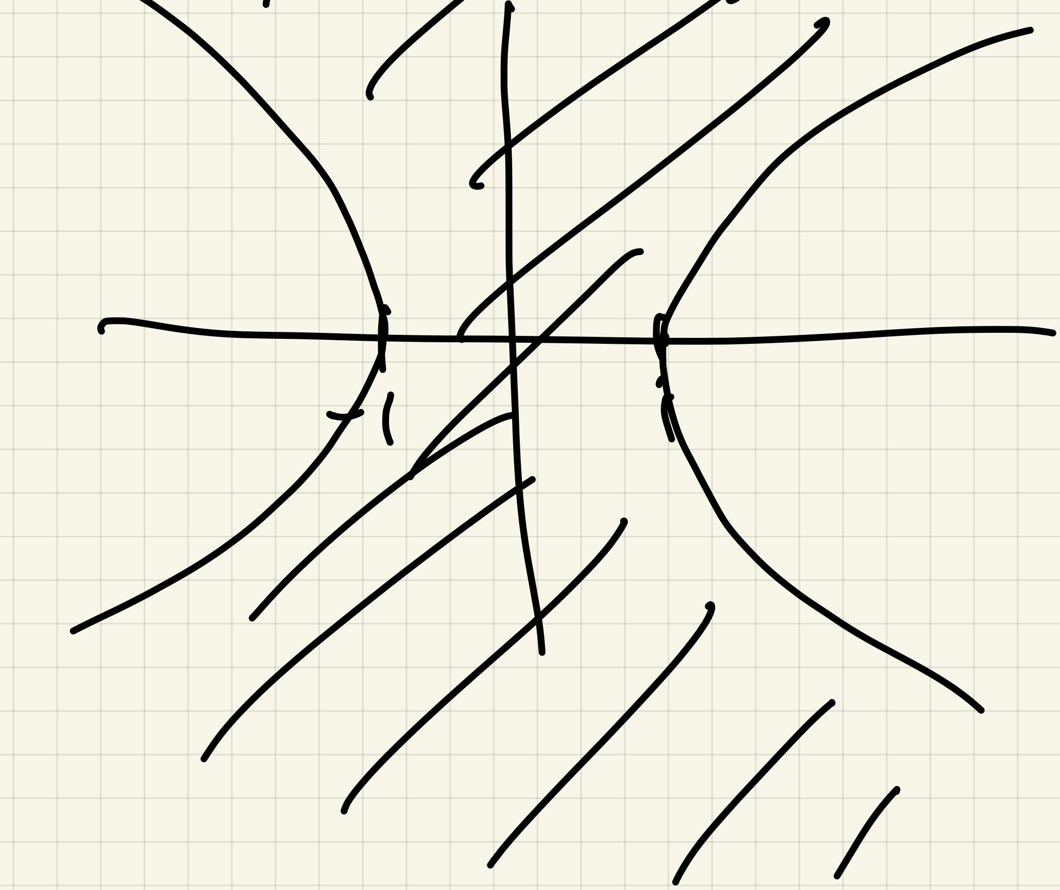
<u>Ex 3</u> (a) $z = x^2 + y^2$



(b) $z = \sqrt{1 + y^2 - x^2}$

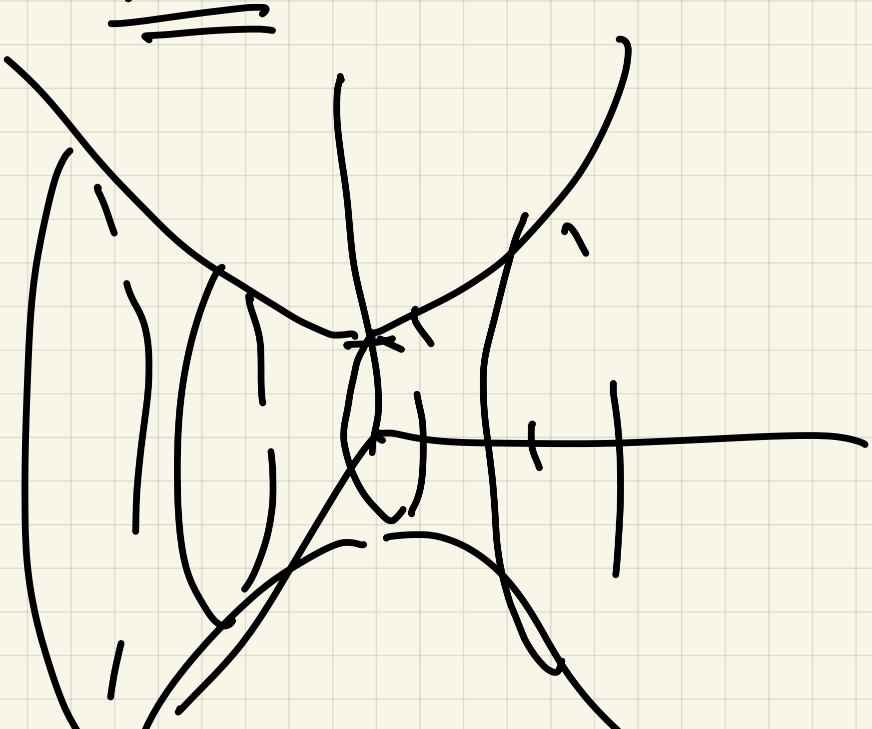
<u>Domain</u>: $1 + y^2 - x^2 \geq 0$

$1 \geq x^2 - y^2$



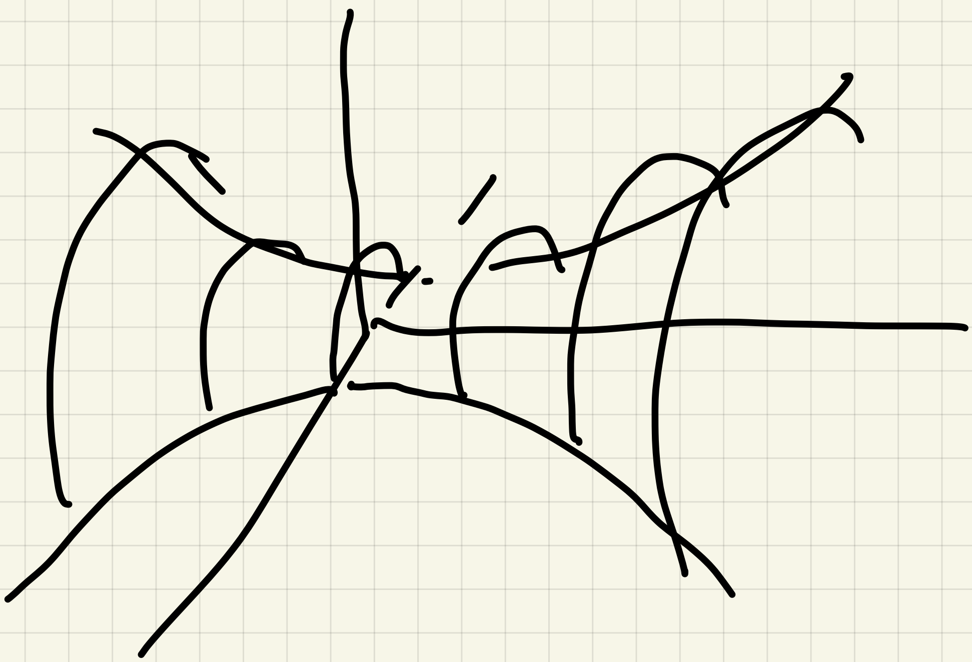
Graph: $z^2 = 1 + y^2 - x^2$

$$\underline{\underline{x^2 + z^2}} = 1 + y^2$$



correct answer

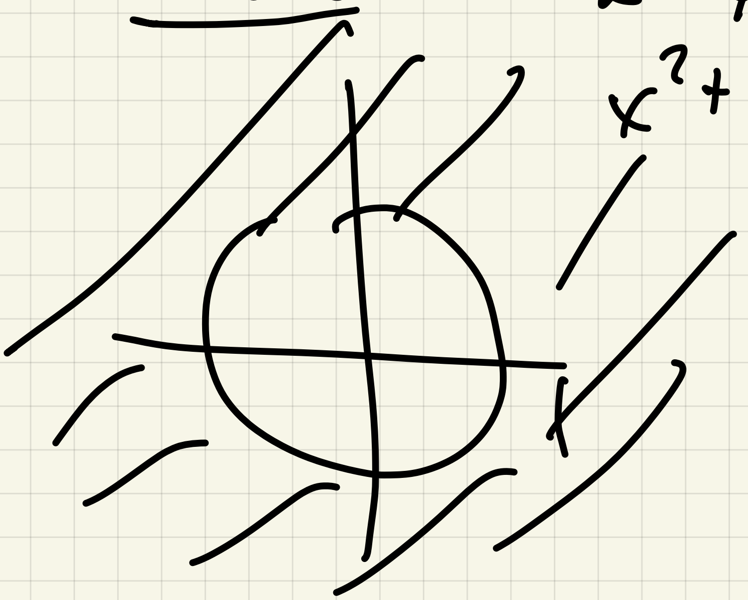
$z \geq 0$



(c) $z = -\sqrt{x^2+y^2-1}$

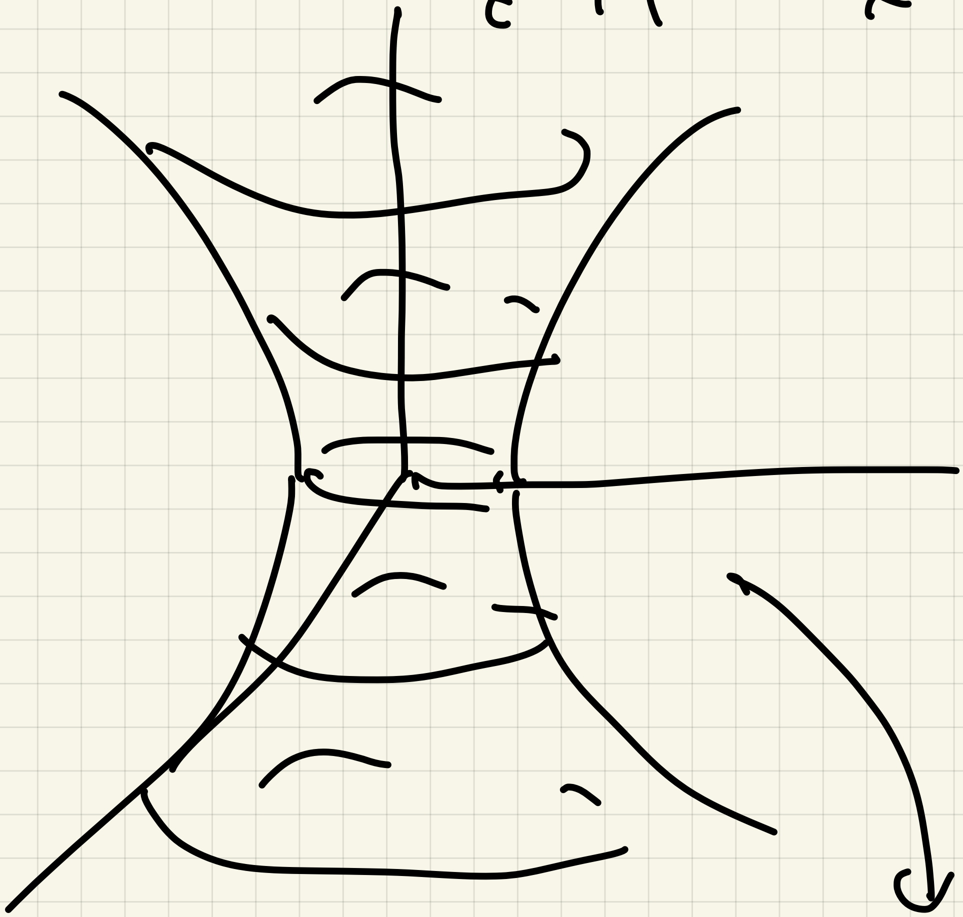
Domain: $x^2+y^2-1 \geq 0$

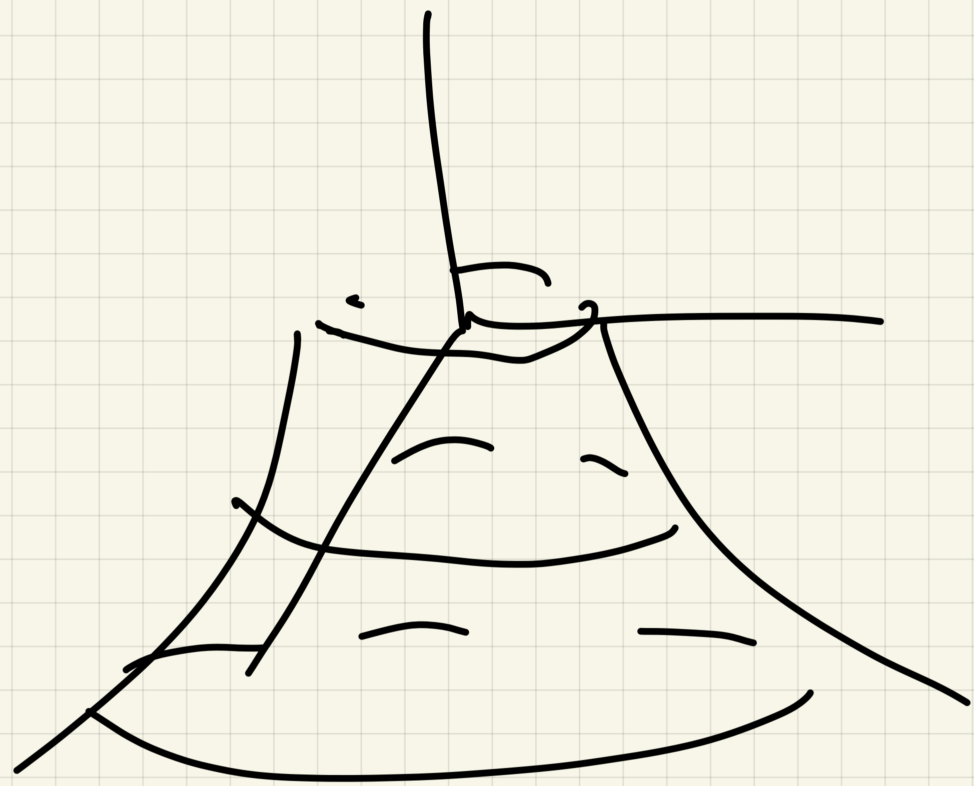
$x^2+y^2 \geq 1$



Graph : $z^2 = x^2 + y^2 - 1$

$z^2 + 1 = x^2 + y^2$





13.2 Limits and Continuity

~~lim~~ $z = f(x,y)$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Calc I

$$\lim_{x\to a} f(x) = L$$

① Easy limits:

Basic: If c is constant,

(a) $$\lim_{(x,y)\to(a,b)} c = c$$

(b) $$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Properties:

Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$

$\lim g(x,y) = M$

Then

1. $\lim \left(f(x,y) + g(x,y)\right) = L + M$
2. $\lim f - g = L - M$
3. $\lim kf = k \cdot L$
   k const
4. $\lim f(x,y)\, g(x,y) = L \cdot M$
5. $\lim \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ if $M \neq 0$
6. $\lim f(x,y)^n = L^n$
7. $\lim \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

   for $n > 0$, $L > 0$ if n even
8. If $h : \mathbb{R} \to \mathbb{R}$ is continuous

Then

$$\lim h(f(x,y)) = h(L)$$

This allows lots of easy limits

Ex (a) $\displaystyle\lim_{(x,y)\to(3,2)} x+y^3 = 3+2^3 = 11$

(PWP 1, 6)

(b) $\displaystyle\lim_{(x,y)\to(3,2)} \frac{x+y^3}{x^2-y^2} = \frac{11}{5}$

(c) $\displaystyle\lim_{(x,y)\to(3,2)} \ln(x^3-y^3) =$

$\ln(3^3-2^3) =$

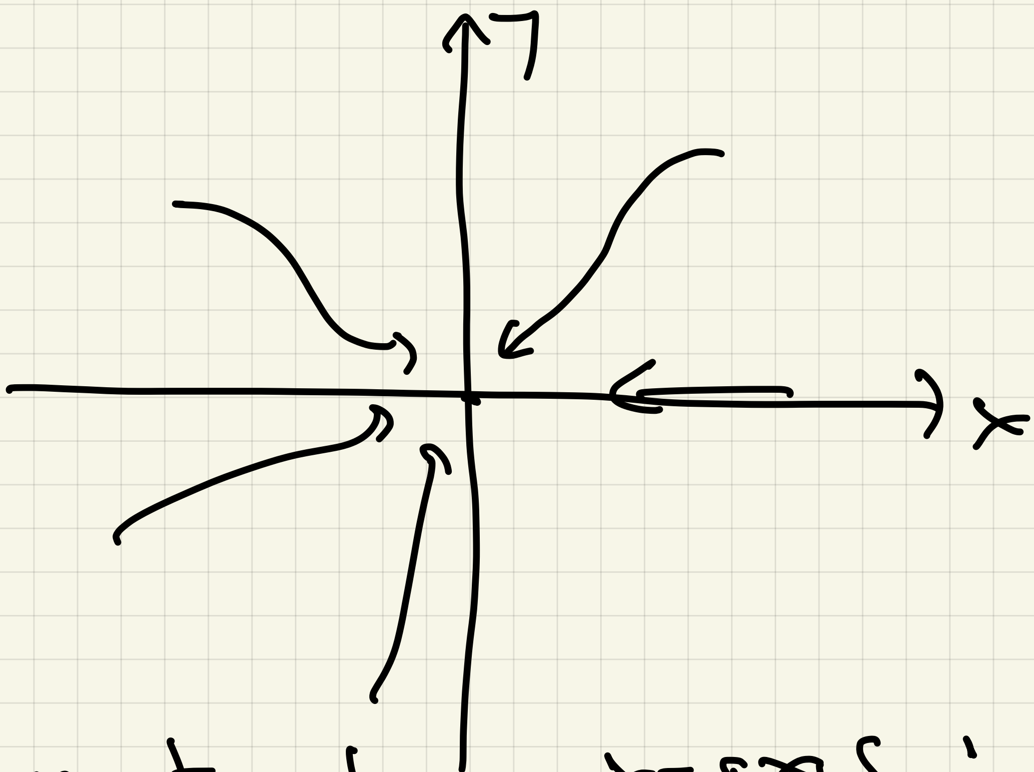
$\ln 19$

(d) $\displaystyle\lim_{(x,y)\to(3,2)} \frac{\ln\left(\arctan\frac{x}{y} + x^2\right)}{\sqrt{\sin y}}$

$$= \frac{\ln(\tan^{-1} 3/2 + 9)}{\sqrt{\sin 2}}$$

② Non obvious limits

Ex2 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x+3y}{x+y}$



limit along x-axis:

(set $y=0$)

$$\lim_{(x,y)\to(0,0)} \frac{2x+3y}{x+y} = \lim_{x\to 0} \frac{2x}{x} = 2$$

ok, b/c $x \neq 0$

x close to 0

$\lim_{(x,y)\to}$ <u>limit along y-axis</u>

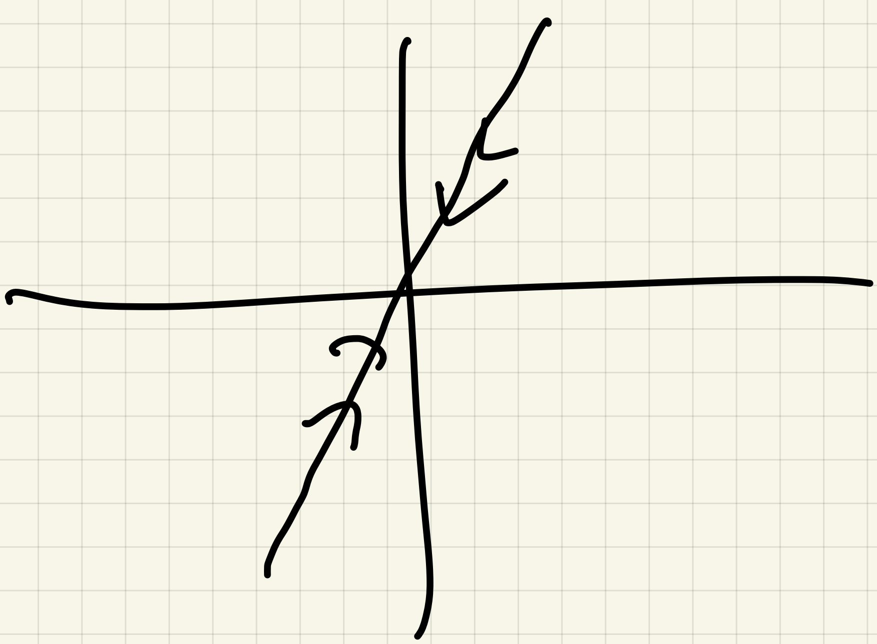
Set $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x+3y}{x+y} = \lim \frac{3y}{y} = 3$$

<u>Conclusion</u> $\lim_{(x,y)\to(0,0)} \frac{2x+3y}{x+y}$ <u>DNE</u>

there's no single limiting value.

<u>Ex</u> What is limit along the line $y = 4x$



$y = 4x$

$$\lim_{(x,y)\to(0,0)} \frac{2x+3y}{x+y} = \lim_{x\to 0} \frac{2x+3(4x)}{x+(4x)} =$$

$$\lim_{x\to 0} \frac{14x}{5x} = \frac{14}{5}$$

Ex 3

$$\lim_{(x,y)\to(0,0)} \frac{3x^2+3y^2+x^4-y^4}{x^2+y^2}$$

Idea : Algebra :

$$= \frac{3(x^2+y^2)+(x^2+y^2)(x^2-y^2)}{x^2+y^2}$$

$(x,y) \neq (0,0)$

$$= \lim_{(x,y)\to(0,0)} \frac{3+(x^2-y^2)}{1} = 3$$