2/11/ Calc3 : Exam 1 Thursday

Last week $\bar{r}(t)$ = vector valued function

$\bar{v}(t) = \bar{r}'(t)$ = velocity

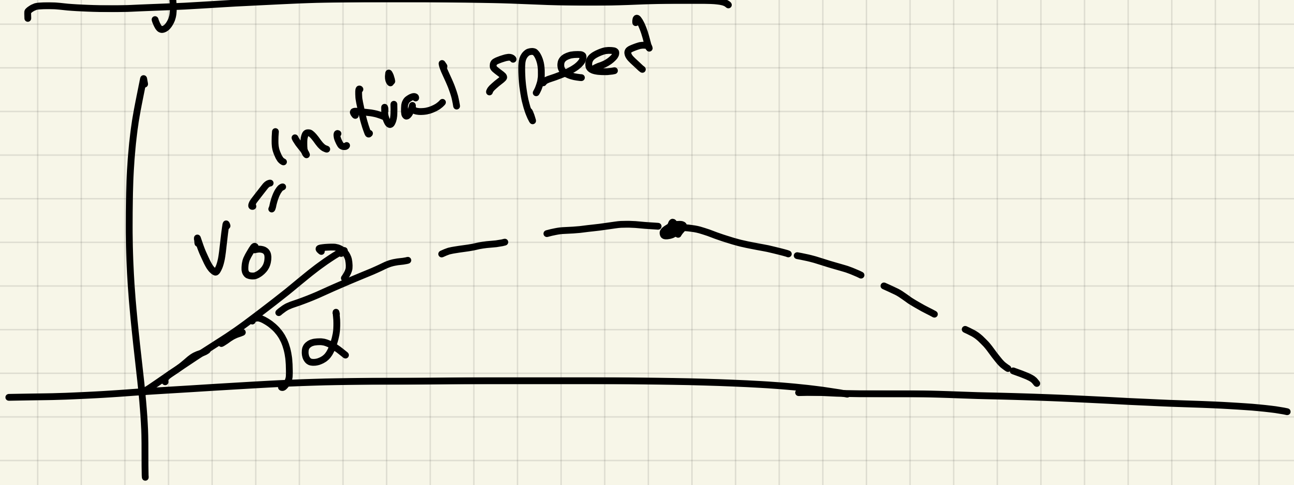
= tangent

$\bar{a}(t) = \bar{r}''(t)$ = acceleration

$|\bar{v}(t)|$ = speed

$\int \bar{r}(t)\,dt \qquad \int_a^b \bar{r}(t)\,dt$

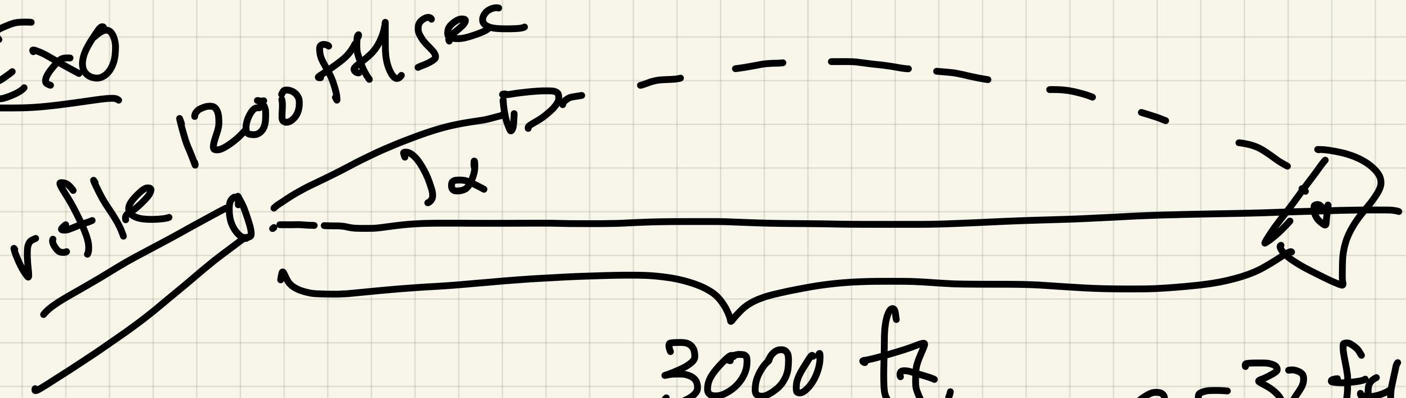
Projectile motion:



$\bar{r}(t) = \langle (v_0\cos\alpha)t, (v_0\sin\alpha)t - \frac{1}{2}gt^2 \rangle$

gravity

Ex0



$g = 32\,ft/s^2$

What angle α to fire at?

$$\vec{r}(t) = \langle \underbrace{(1200\cos\alpha)t}_{x}, \underbrace{(1200\sin\alpha)t - 16t^2}_{y} \rangle$$

$$y = 0 \Rightarrow t = 0, \frac{1200\sin\alpha}{16}$$

Want $x = 3000$ when $y = 0$, i.e. $t = \frac{1200\sin\alpha}{16}$

so $3000 = x = (1200\cos\alpha)\left(\frac{1200\sin\alpha}{16}\right)$

$$3000 = 75 \cdot 1200 \cos\alpha\sin\alpha$$

$$\frac{1}{30} = \frac{5}{75 \cdot 2} = \frac{3000}{75 \cdot 1200} = \cos\alpha\sin\alpha$$

so $\frac{1}{15} = \frac{2}{30} = 2\cos\alpha\sin\alpha = \sin 2\alpha$

$\therefore\ 2\alpha = \arcsin\frac{1}{15} \Rightarrow$

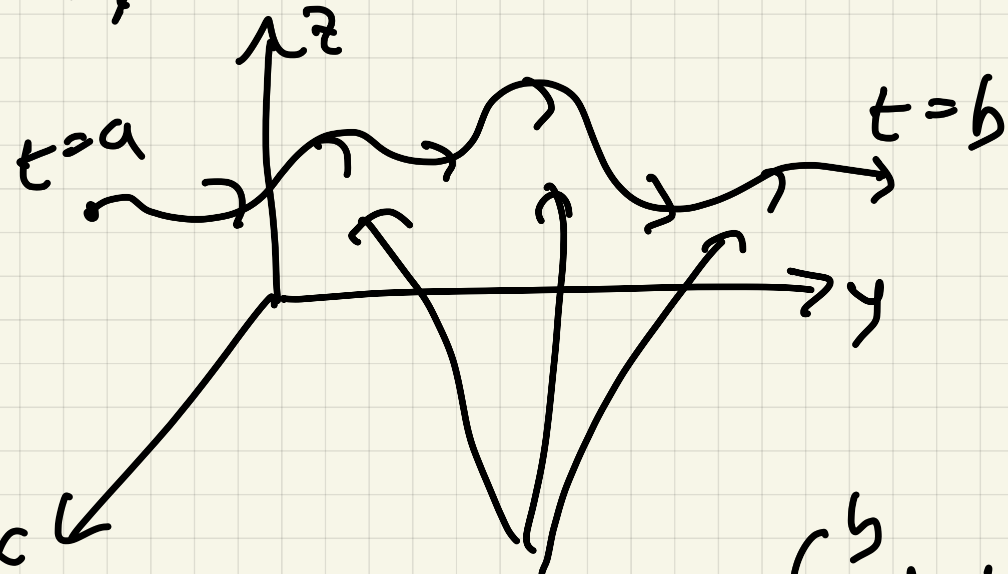
$2\alpha = 3.82^\circ \Rightarrow \alpha = 1.91^\circ$

$\left(\text{Also } 2\alpha = 180^\circ - 3.82 \Rightarrow \alpha = 90 - 1.91 = 88.09^\circ\right)$

§ 12.3 & 12.4

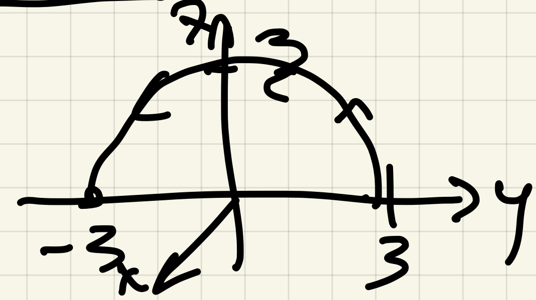
(will see this in Ch. 15 later)

If a curve C in $\mathbb{R}^3$ is traced once by $\vec{r}(t)$ for $a \le t \le b$,



Then Arclength = length $= \int_a^b |\vec{r}\,'(t)|\,dt$

Ex



$$\bar{r}(t) = \langle 0, 3\cos t, 3\sin t\rangle, \quad 0 \le t \le \pi$$

$$\text{Arclength} = \int_0^{\pi} |\bar{r}'(t)|\,dt = \int_0^{\pi} \underbrace{|\langle 0, -3\sin t, 3\cos t\rangle|}_{3}\,dt$$

$$= \int_0^{\pi} 3\,dt = 3\pi$$

[Check: Circle radius 3 has circumference $2\pi r = 2\pi \cdot 3 = 6\pi$, so half circle is 3π ✓]

Ex 2 $\bar{r}(t) = \langle \underbrace{2+\sin t}_{x}, \underbrace{2-\sin t}_{y}, \underbrace{\sqrt{2}\cos t}_{z}\rangle \quad 0 \le t \le 2\pi$

$$\bar{v}(t) = \bar{r}'(t) = \langle \cos t, -\cos t, -\sqrt{2}\sin t\rangle$$

$$|\bar{v}(t)| = \sqrt{\cos^2 t + \cos^2 t + 2\sin^2 t} = \sqrt{2}$$

so $\text{Length} = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$

Can visualize curve:

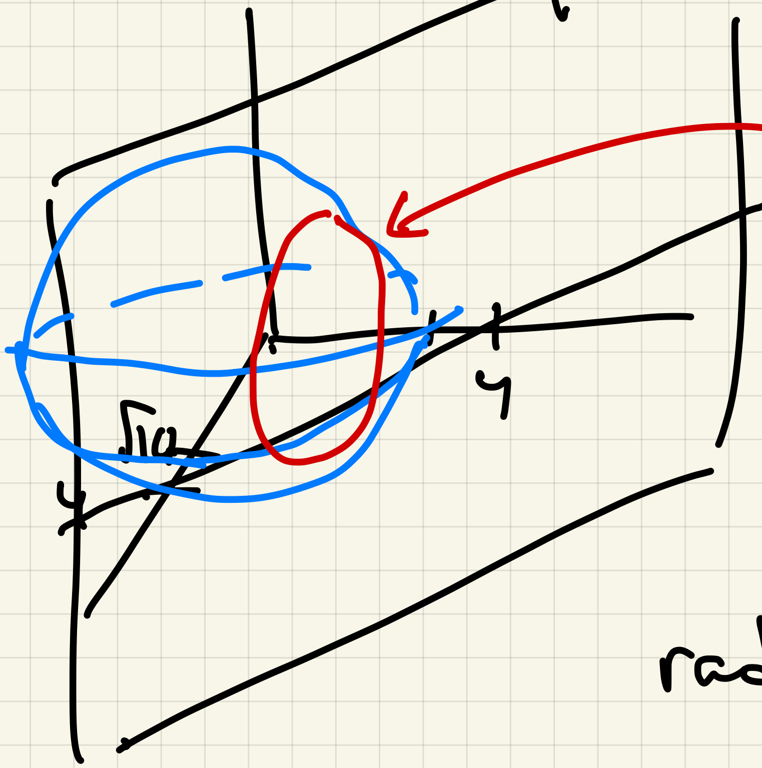
Observe: $x + y = 4$

$$x^2 + y^2 + z^2 =$$

$$4 + \cancel{4\sin t} + \sin^2 t + 4 - \cancel{4\sin t} + \sin^2 t + 2\cos^2 t$$

$$= 8 + 2 = 10$$

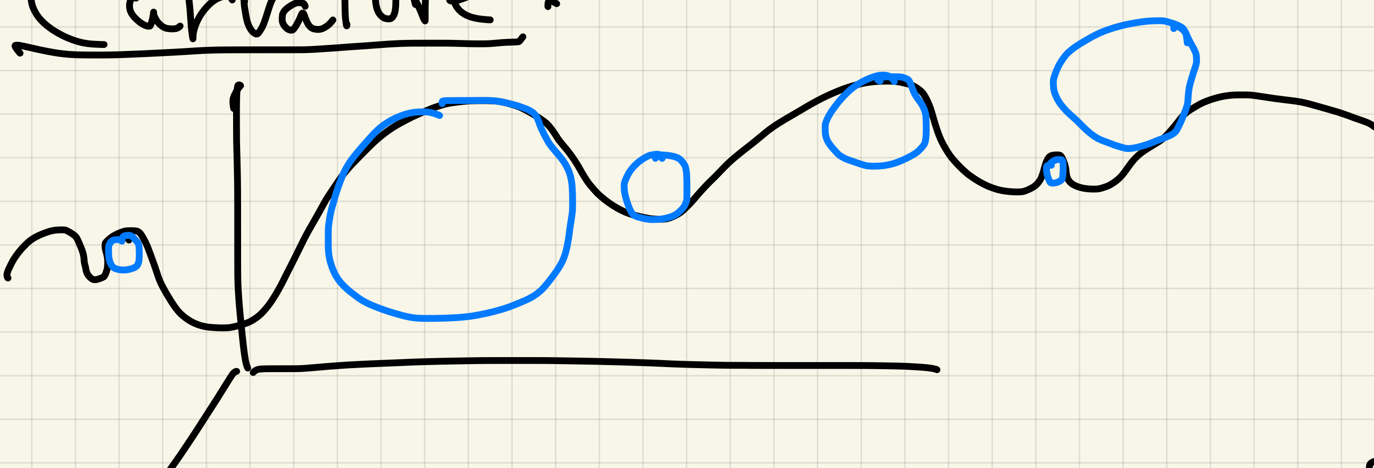
So curve is intersection of plane $x+y=4$ and sphere $x^2+y^2+z^2=10$



Curve is red circle

Since Length $= 2\sqrt{2}\pi$ is circumference, radius is $\sqrt{2}$.

Curvature:



Given $\vec{r}(t)$, how tight are the turns? What is the radius R of the best-fit circle at each point?

<u>Answer</u>: let $T(t) = \frac{\bar{v}(t)}{|\bar{v}(t)|}$ be the <u>unit tangent</u> vector to $\bar{r}(t)$.

Then best fit radius R is

$$R = \frac{1}{K}, \text{ where } K = \frac{|T'(t)|}{|\bar{v}(t)|}$$

Curvature

<u>Ex3</u> Find curvature of red circle in Ex2 at $t = \pi/2$.

<u>Easy</u>: We already found that $|\bar{v}(t)| = \sqrt{2}$, so

$$T(t) = \frac{\bar{v}(t)}{|\bar{v}(t)|} = \frac{\langle \cos t, -\cos t, -\sqrt{2}\sin t\rangle}{\sqrt{2}} \Rightarrow$$

$$T'(t) = \frac{\langle -\sin t, \sin t, -\sqrt{2}\cos t\rangle}{\sqrt{2}} \Rightarrow$$

$$|T'(t)| = \frac{\sqrt{\sin^2 t + \sin^2 t + 2\cos^2 t}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\therefore \quad K = \frac{|T'(t)|}{|\bar{v}(t)|} = \frac{1}{\sqrt{2}} \Rightarrow$$

R = radius of curvature is $\sqrt{2}$

But usually computing $T'(t)$ is brutal:

<u>Ex 4</u> $\quad \bar{r}(t) = \langle \sqrt{4-t^2}, t, 4-t^2 \rangle$

$$\bar{r}'(t) = \left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle$$

$$T(t) = \frac{\left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle}{\sqrt{\frac{t^2}{4-t^2} + 1 + 4t^2}}$$

$T'(t) = ??$ A mess!

Thankfully, there's an easier way:

Thm $$K = \frac{|\bar{v} \times \bar{a}|}{|\bar{v}|^3} = \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|^3}$$

Back to Ex 4 Find curvature at $t = 0$.

$$\bar{v}(t) = \left\langle \frac{-t}{\sqrt{4-t^2}}, 1, -2t \right\rangle \Rightarrow \bar{v}(0) \langle 0, 1, 0 \rangle$$

$$\bar{a}(t) = \left\langle \frac{-1}{\sqrt{4-t^2}} + \frac{t^2}{(4-t^2)^{3/2}}, 0, -2 \right\rangle \Rightarrow$$

$$\bar{a}(0) = \left\langle -\frac{1}{2}, 0, -2 \right\rangle$$

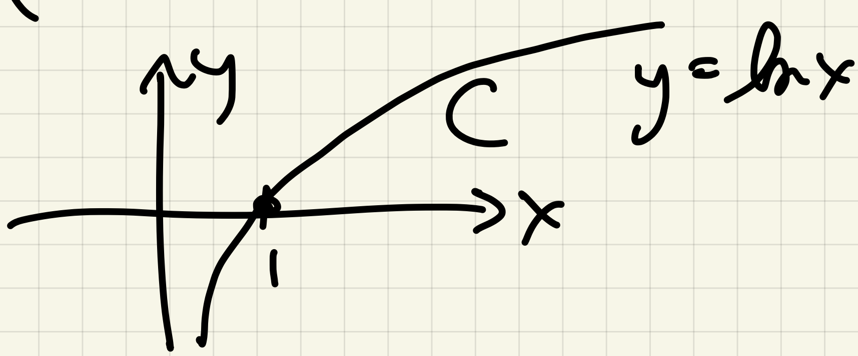
$$\bar{v}(0) \times \bar{a}(0) = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & -2 \end{vmatrix} = \left\langle -2, 0, \frac{1}{2} \right\rangle,$$

so

$$K = \frac{|\bar{v}(0) \times \bar{a}(0)|}{|\bar{v}(0)|^3} = \frac{\left|\left\langle -2, 0, \frac{1}{2} \right\rangle\right|}{|\langle 0, 1, 0 \rangle|^3} = \frac{\sqrt{\frac{17}{4}}}{1} = \frac{\sqrt{17}}{2}$$

$$\left(\text{so radius curvature} = \frac{2}{\sqrt{17}}\right)$$

Ex 5



(a) Find curvature at $(1,0)$
(b) Where is curvature largest?

Parametrize C : $\bar{r}(t) = \langle \underset{x}{e^t}, \underset{y}{t}, \underset{z}{0} \rangle$

$(y: t = \ln e^t = \ln x)$ ✓

$\bar{v}(t) = \bar{r}'(t) = \langle e^t, 1, 0 \rangle$

$\bar{a}(t) = \langle e^t, 0, 0 \rangle$

$|\bar{v}(t)| = \sqrt{e^{2t}+1}$

$$v \times a = \begin{vmatrix} i & j & k \\ e^t & 1 & 0 \\ e^t & 0 & 0 \end{vmatrix} = \langle 0, 0, -e^t \rangle$$

$$\text{so } K(t) = \frac{|\langle 0,0,-e^t \rangle|}{|\langle e^t,1,0 \rangle|^3} = \frac{e^t}{(e^{2t}+1)^{3/2}}$$

(a) $\bar{r}(0) = \langle 1,0,0 \rangle \sim (1,0)$, so

Curvature at $(1,0)$ is at $t=0$,

$$\therefore K(0) = \frac{e^0}{(1+1)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{\sqrt{8}},$$

radius of curvature is $\sqrt{8} = R$,

(b) <u>Maximize</u> $K(t) = \dfrac{e^t}{(e^{2t}+1)^{3/2}}$

$$\frac{dK}{dt} = \frac{(e^{2t}+1)^{3/2} e^t - e^t \cdot \frac{3}{2}(e^{2t}+1)^{1/2} \cdot 2e^{2t}}{(e^{2t}+1)^3}$$

$$= \frac{e^t(e^{2t}+1)^{1/2}(e^{2t}+1-3e^{2t})}{(e^{2t}+1)^3} = 0$$

$$\Rightarrow 0 = e^{2t}+1-3e^{2t} = 1-2e^{2t} \Rightarrow$$

$$2e^{2t} = 1 \Rightarrow e^{2t} = \frac{1}{2} \Rightarrow 2t = \ln\frac{1}{2} \Rightarrow$$

$$t = \frac{1}{2}\ln\frac{1}{2} = -\frac{1}{2}\ln 2 \approx -.347$$

so Max curvature is

$$K\left(\frac{1}{2}\ln\frac{1}{2}\right) = K\left(\ln\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(\left(\frac{1}{\sqrt{2}}\right)^2+1\right)^{3/2}}$$

$$= \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{3/2}} = \frac{\frac{1}{\cancel{\sqrt{2}}}}{\frac{3\sqrt{3}}{2\cancel{\sqrt{2}}}} = \frac{2}{3\sqrt{3}} \quad \left(R = \frac{3\sqrt{3}}{2}\right)$$