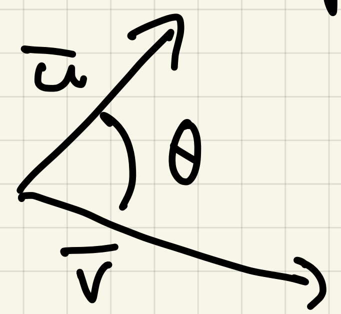
8/28 / Calc3

dot product $\bar{u} \cdot \bar{v}$

Last time



$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{|\bar{u}||\bar{v}|} \leftarrow$$

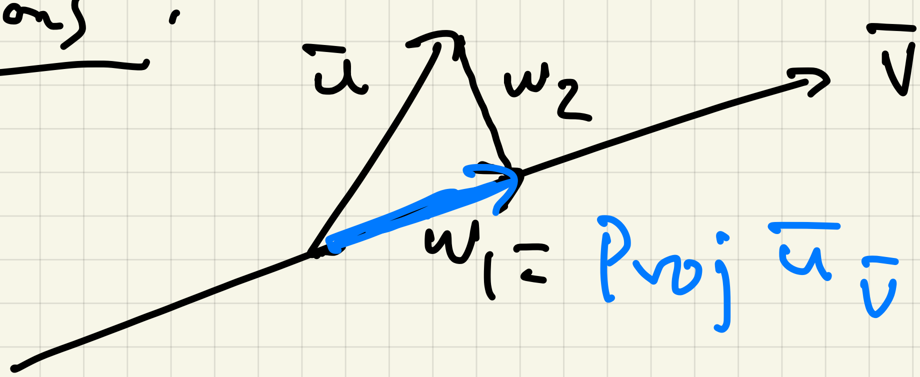
$\bar{u} \cdot \bar{v} = 0 \Rightarrow \theta = 90^\circ$

perpendicular
normal
orthogonal

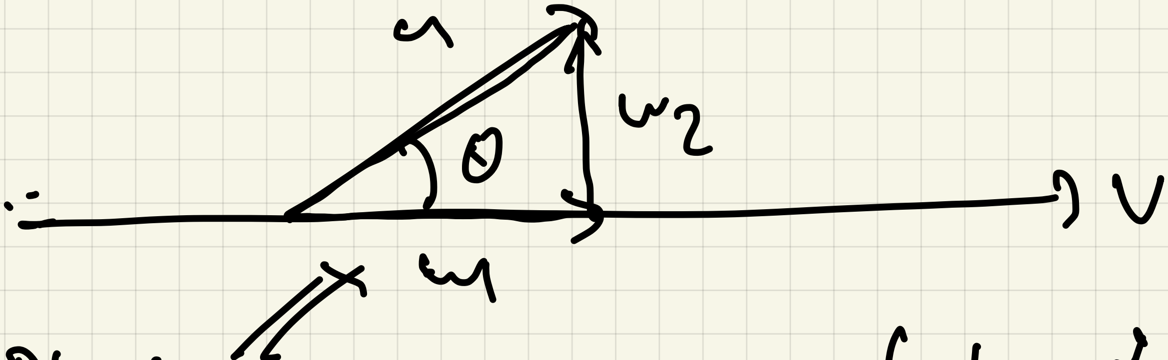
$\bar{u} \cdot \bar{v} > 0 \Rightarrow \theta < 90^\circ$

$\bar{u} \cdot \bar{v} < 0 \Rightarrow \theta > 90^\circ$

Projections:



Idea: Want to decompose $\bar{u}$ into $\bar{w}_1 + \bar{w}_2$, $\bar{w}_1 \parallel \bar{v}$ and $w_2 \perp \bar{v}$



Picture $\Rightarrow$ w_1 has $\begin{cases} \text{direction } \bar{v} \\ \text{length } |u|\cos\theta \end{cases}$

so $\bar{w}_1 = |u| \cos\theta \dfrac{\bar{v}}{|\bar{v}|}$

$= \cancel{|u|} \dfrac{\bar{u}\cdot\bar{v}}{\cancel{|u|}|\bar{v}|} \dfrac{\bar{v}}{|\bar{v}|} = \dfrac{\bar{u}\cdot\bar{v}}{\underbrace{\bar{v}\cdot\bar{v}}_{=|\bar{v}|^2}} \bar{v}$

$\begin{cases} \underline{\text{Thm}} \quad \text{Proj}_{\bar{v}}\bar{u} = \dfrac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}} \bar{v} \\ \qquad w_2 = u - w_1 \end{cases}$

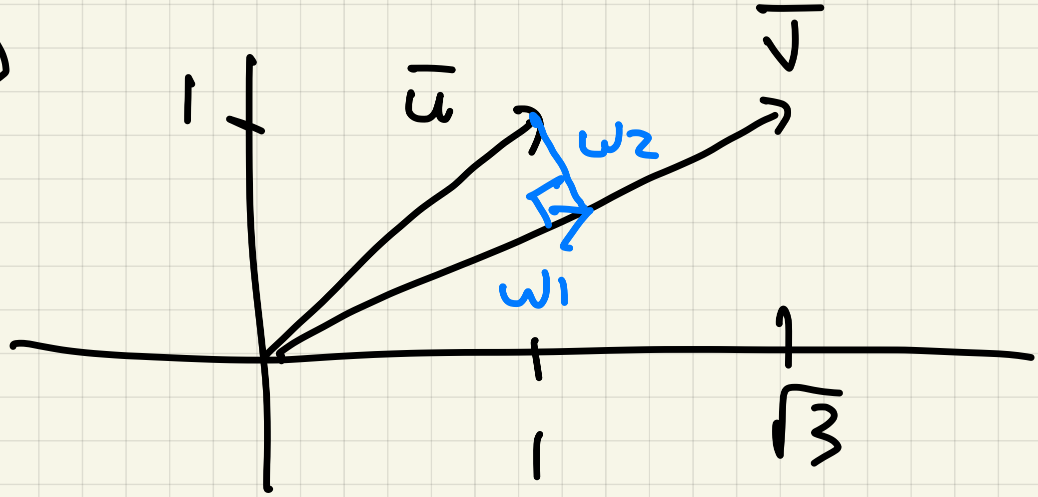
(The scalar component of $\bar{u}$ in direction $\bar{v}$ is $= \dfrac{\bar{u}\cdot\bar{v}}{|v|}$

$|u|\cos\theta$)

Ex) Find vector components of $\bar{u}$ along $\bar{v}$ and orthogonal to $\bar{v}$

$w_1 = \text{Proj}_{\bar{v}} \bar{u}$; w_2

(a)



$$\bar{u} = \langle 1, 1 \rangle, \quad \bar{v} = \langle \sqrt{3}, 1 \rangle$$

$$w_1 = \text{Proj}_{\bar{v}} \bar{u} = \frac{\bar{u} \cdot \bar{v}}{\bar{v} \cdot \bar{v}} \bar{v}$$

$$= \frac{(\sqrt{3}+1)}{4} \langle \sqrt{3}, 1 \rangle$$

$$= \left\langle \frac{3+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right\rangle$$

$$w_2 = \bar{u} - \bar{w}_1$$

$$\langle 1,1\rangle - \left\langle \frac{3+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4} \right\rangle =$$

$$\left\langle \frac{1-\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4} \right\rangle$$

(scalar component $\frac{\sqrt{3}+1}{2}$)

(b) $\bar{u} = \langle 3, -2, 7\rangle$

(ii) $\bar{v} = \hat{i} = \langle 1, 0, 0\rangle$

$$\bar{w}_1 = \frac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}} v = \frac{3}{1}\langle 1, 0, 0\rangle =$$

$$\langle 3, 0, 0\rangle$$

$$w_2 = \langle 0, -2, 7\rangle$$

scalar component $= 3$

(iii) $v = \hat{j} = \langle 0, 1, 0\rangle$

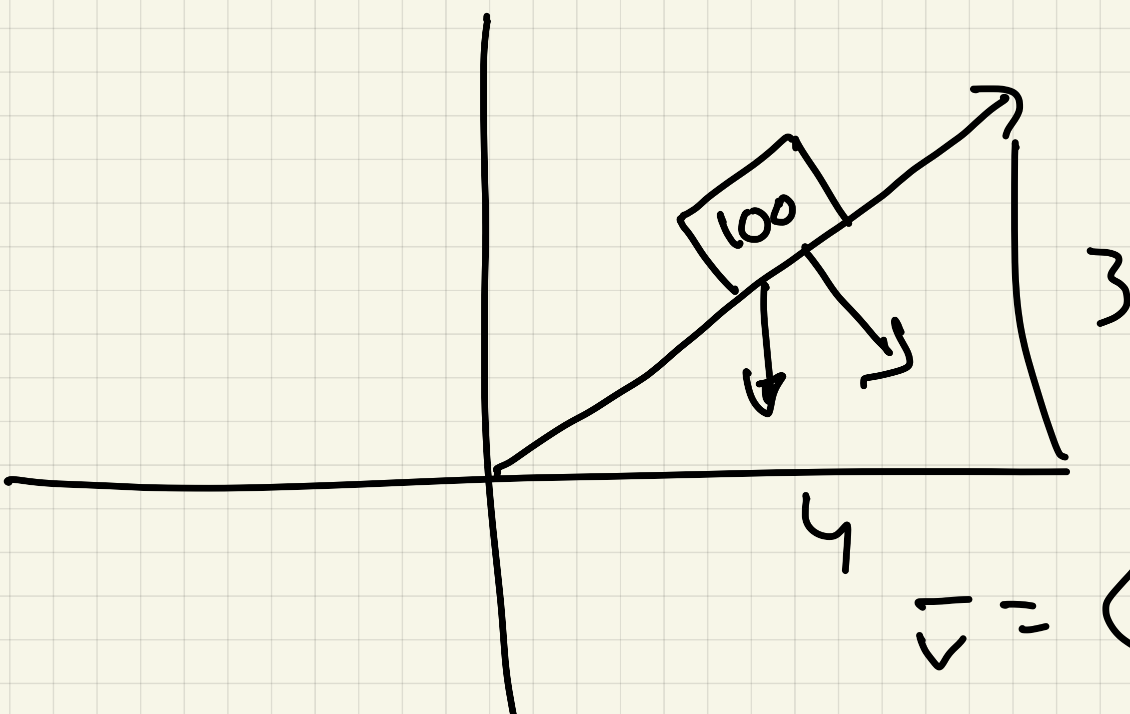
$$w_1 = \frac{\bar{u}\cdot\bar{v}}{\bar{v}\cdot\bar{v}}\bar{v} = \frac{-2}{1}\langle 0, 1, 0\rangle$$

$$= \langle 0, -2, 0\rangle$$

$w_2 = \langle 3, 0, 7 \rangle$

scalar component $= -2$

Ex2 A 100 lb box sits on a ramp with slope $3/4 = m$. Find components of gravity along/orthogonal to ramp



$\bar{v} = \langle 4, 3 \rangle$

(only direction matters)

$\bar{u} = \text{gravity} = \langle 0, -100 \rangle$

Component along v = $\text{Proj}_{\bar v}\bar u$ =

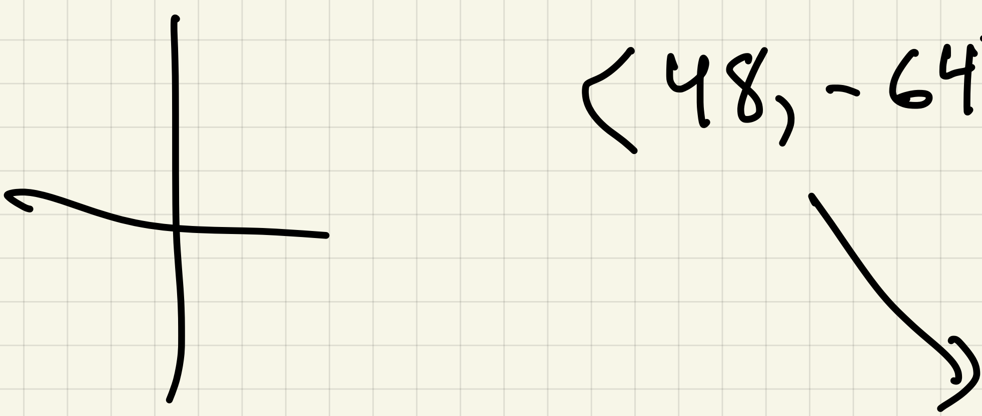
$$\frac{\bar u\cdot \bar v}{\bar v\cdot \bar v} v = \frac{\langle 0,-100\rangle\cdot\langle 4,3\rangle}{\langle 4,3\rangle\cdot\langle 4,3\rangle}\cdot\langle 4,3\rangle$$

$$\frac{-300}{25}\langle 4,3\rangle = -12\langle 4,3\rangle$$

$$= \langle -48,-36\rangle$$

Component orthogonal
$=$
$\bar u - \text{Proj}_{\bar v}\bar u =$
$\langle 0,-100\rangle - \langle -48,-36\rangle =$

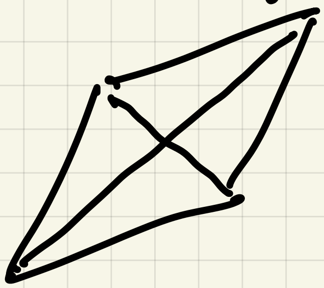
$\langle 48,-64\rangle$

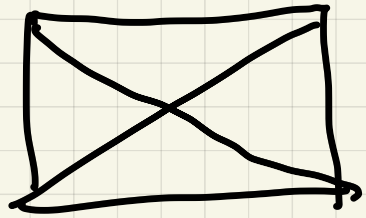


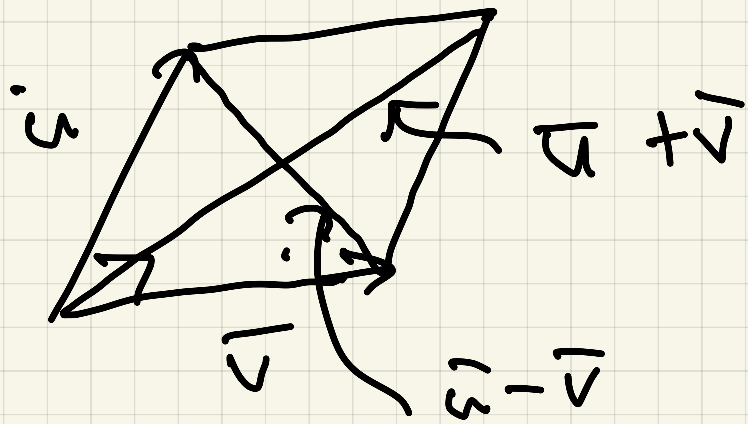
Ex 3 (Geometry)

Show A parallelogram is a rectangle $\Longleftrightarrow$

diagonals have same length







diagonals have same length

$$\Updownarrow$$

$$|\bar{u}-\bar{v}| = |\bar{u}+\bar{v}|$$

$$\Updownarrow$$

$$|\bar{u}-\bar{v}|^2 = |\bar{u}+\bar{v}|^2$$

$$\cancel{\bar{u}\cdot\bar{u}} - 2\bar{u}\cdot\bar{v} + \cancel{\bar{v}\cdot\bar{v}} = \cancel{\bar{u}\cdot\bar{u}} + 2\bar{u}\cdot\bar{v} + \cancel{\bar{v}\cdot\bar{v}}$$

$$-2\bar{u}\cdot\bar{v} = 2\bar{u}\cdot\bar{v}$$

$$\Updownarrow$$

$$4\bar{u}\cdot\bar{v} = 0$$

$$\Updownarrow$$

$$\Leftrightarrow \bar{u}\cdot\bar{v} = 0$$

rectangle

§11.4 If $\bar{u} = \langle u_1, u_2, u_3 \rangle$

$\bar{v} = \langle v_1, v_2, v_3 \rangle$

Then the <u>cross product</u> is

$$\bar{u} \times \bar{v} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

Easy to remember using <u>determinants</u>:

2x2 array of numbers

<u>2x2 determinant</u>:

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \left(= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

<u>Ex1</u> (a) $\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 11$

$$\begin{vmatrix} 1 & 6 \\ 2 & 9 \end{vmatrix} = 9 - 12 = -3$$

3x3 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

minus

Ex2 (a)

$$\begin{vmatrix} 2 & 1 & 3 \\ 7 & 1 & 4 \\ 2 & 6 & 5 \end{vmatrix} =$$

$$2\begin{vmatrix} 1 & 4 \\ 6 & 5 \end{vmatrix} - 1\begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} 7 & 1 \\ 2 & 6 \end{vmatrix}$$

$$2(-19) - (27) + 3(40)$$

$$= -38 - 27 + 120 = 55$$

(b) $\begin{vmatrix} 2 & 3 & 7 \\ 4 & -1 & 6 \\ 2 & 0 & 5 \end{vmatrix} =$

$$2\begin{vmatrix} -1 & 6 \\ 0 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix} + 7\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix}$$

$$-10 - 24 + 14 = -20$$

Now cross-product becomes easy:

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$i(u_1v_3 - u_3v_1) - j(u_1v_3 - u_3v_1)$$
$$+k(u_1v_3 - u_3v_1)$$

<u>Ex 3</u> $\bar{u} = \langle 1, 2, 3 \rangle$
$\bar{v} = \langle 1, 0, 2 \rangle$

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$i \cdot 4 - j(2-3) + k(0-2)$$

$$\langle 4, 1, -2 \rangle$$

$$\bar{v} \times \bar{u} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \langle -4, -1, 2 \rangle$$

$$\bar{u} \times \bar{u} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \bar{0} \; !$$