8/22/ Calc3

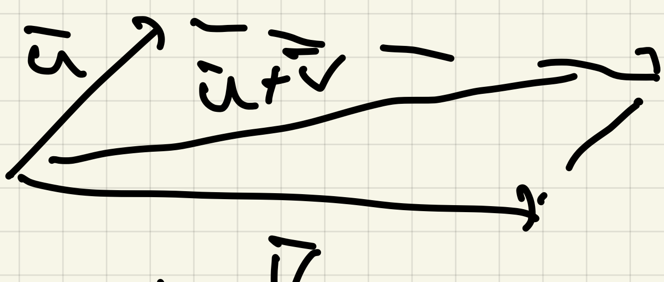
last time: Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

length

component form

Operations $\begin{cases} \bar{u}+\bar{v} & \text{add} \\ k\cdot\bar{u} & k\in\mathbb{R} \text{ scalar mult} \end{cases}$

algebraically, geometrically



Unit vectors

Standard unit vectors

$\hat{i}\ \hat{j}\ \hat{k}$

$\hat{i} = \langle 1,0,0\rangle$

$\hat{j} = \langle 0,1,0\rangle$

$\hat{k} = \langle 0,0,1\rangle$

Decomposition If $\bar{v} \neq \bar{0}$, ① $\frac{\bar{v}}{|\bar{v}|}$ = unit vector

$$\boxed{② \quad \bar{v} = \underbrace{|\bar{v}|}_{\text{length}} \cdot \underbrace{\frac{\bar{v}}{|\bar{v}|}}_{\text{direction}}}$$

Ex | $\bar{v} = -2i + 7j$ in $\mathbb{R}^2$

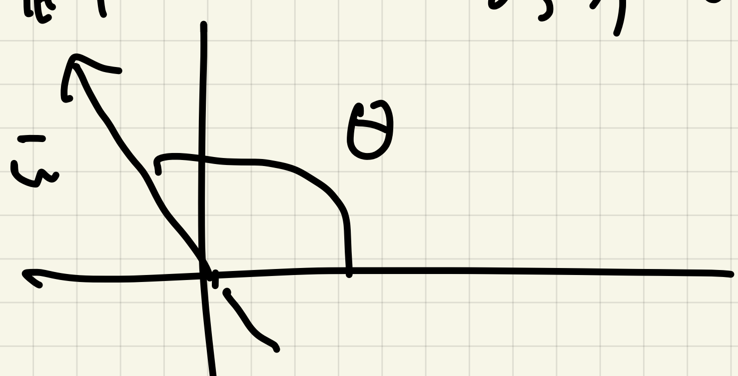
$\quad = -2\langle 1,0\rangle + 7\langle 0,1\rangle = \langle -2, 7\rangle$

Decompose into length / direction

What angle does $\bar{v}$ make with pos x-axis

$|\bar{v}| = \sqrt{(-2)^2 + 7^2} = \sqrt{53}$ length

$\bar{u} = \frac{v}{\|\bar{v}\|} = \left\langle \frac{-2}{\sqrt{53}}, \frac{7}{\sqrt{53}} \right\rangle$ direction



$\theta = \arctan\left(\frac{7}{-2}\right)$ ×

fix this :

$\theta = \pi + \arctan\left(\frac{7}{-2}\right)$

Easier :

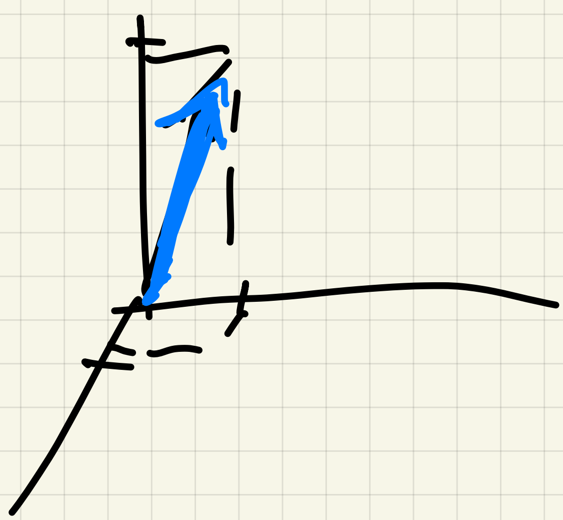
$\theta = \arccos\left(\frac{-2}{\sqrt{53}}\right)$

(5) $\bar{v} = \langle 2, 3, 6 \rangle$

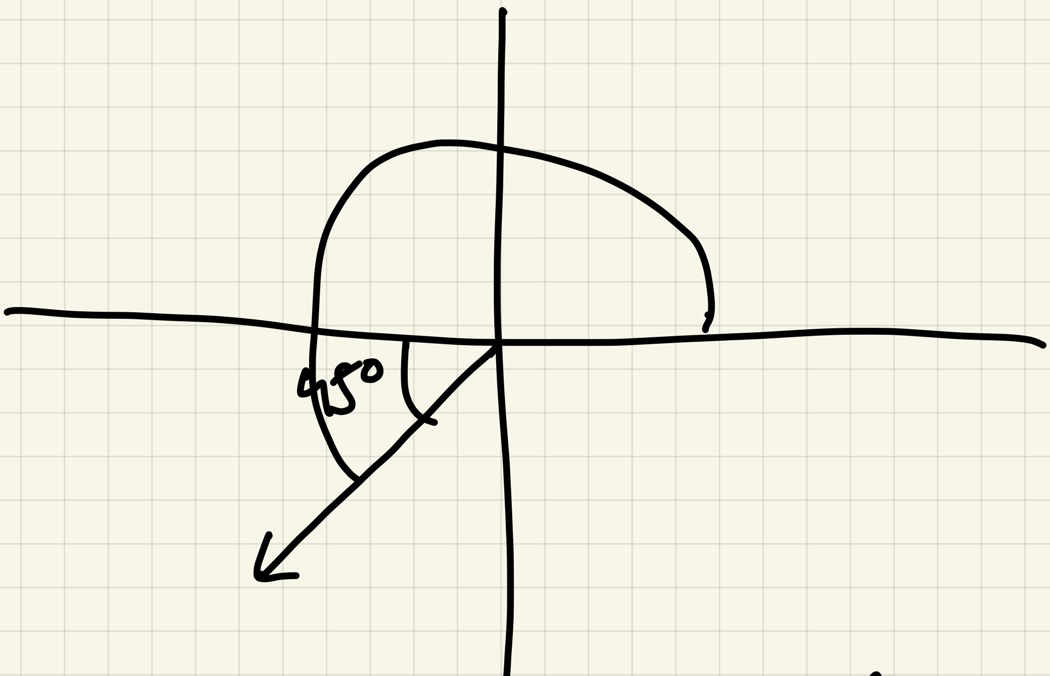
$|\bar{v}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

length

direction : $\bar{u} = \frac{\bar{v}}{7} = \langle 2/7, 3/7, 6/7 \rangle$



Ex2 Find the vector of length $\sqrt{8}$ making a 225° angle with $\left(\frac{5\pi}{4} \text{ rads}\right)$ pos x-axis



direction: $\bar{u} = \langle \cos\frac{5\pi}{4}, \sin\frac{5\pi}{4} \rangle$

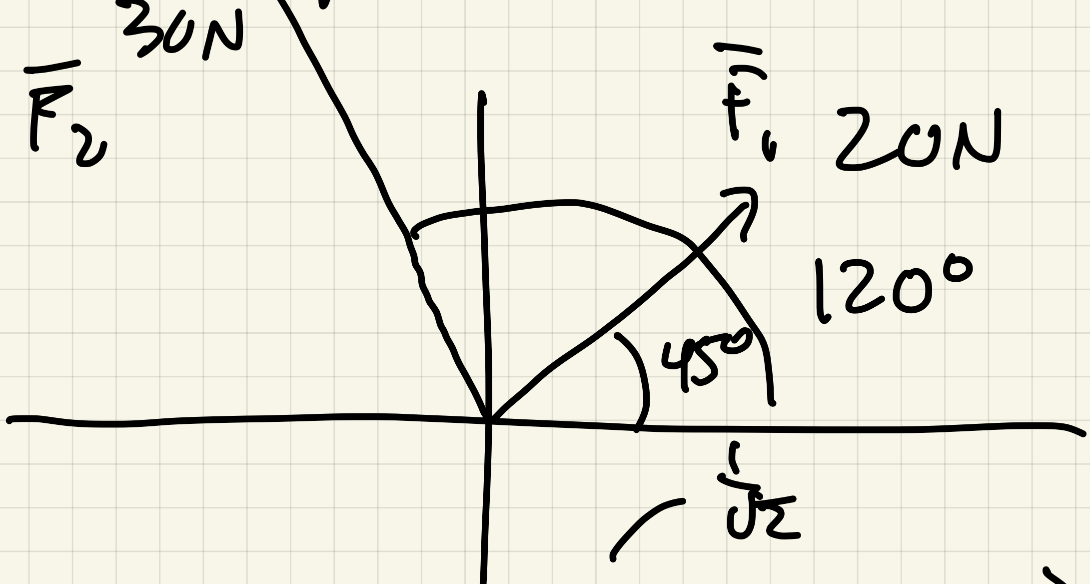
$= \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$\bar{v} = \bar{u} \cdot \sqrt{8} =$

$\sqrt{8}\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \langle -2, -2 \rangle$

Vectors: useful for forces in Physics & Engin.

Ex) If we combine the forces shown, what is the net force's magnitude & direction?



$$\overline{F}_1 = 20\langle \cos 45^\circ, \sin 45^\circ \rangle$$

$$= \left\langle \frac{20}{\sqrt{2}}, \frac{20}{\sqrt{2}} \right\rangle$$

$$\overline{F}_2 = 30\langle \cos 120^\circ, \sin 120^\circ \rangle$$

$$= 30\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

so $\overline{F}_{net} = \overline{F}_1 + \overline{F}_2 =$

$$\left\langle \frac{20}{\sqrt{2}} - 15, \frac{20}{\sqrt{2}} + \frac{30\sqrt{3}}{2} \right\rangle =$$

$$\left\langle \frac{20\sqrt{2}-30}{2}, \frac{20\sqrt{2}+30\sqrt{3}}{2} \right\rangle =$$

$$\langle 10\sqrt{2}-15, 10\sqrt{2}+15\sqrt{3}\rangle =$$

$$\approx \langle -.858, 40.123\rangle$$

$$\|F_{net}\| = 40.132$$

direction $\theta = \arctan\left(\frac{10\sqrt{2}+15\sqrt{3}}{10\sqrt{2}-15}\right) =$

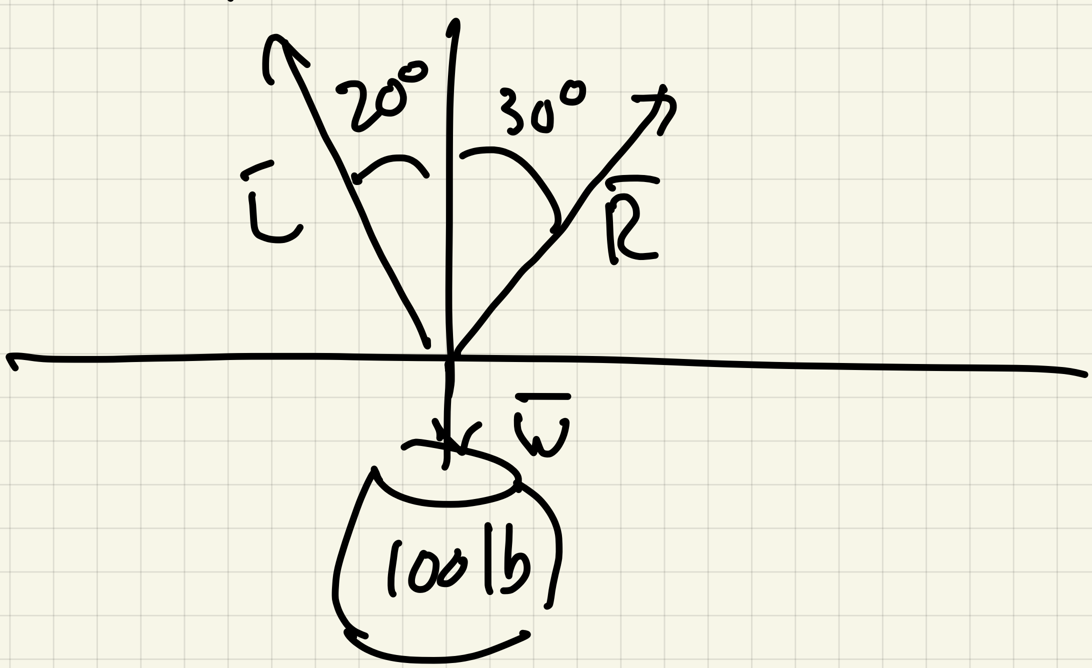
-1.5499 rads

$= -88.78^\circ$

$\swarrow$ ref angle

91.22°

Ex2 Two workers hold up 100 lb wt as shown:
How much force does each lift with?



Key

$$\bar{L} + \bar{R} + \bar{w} = \bar{0}$$

$$\bar{w} = \langle 0, -100 \rangle$$

$$L\langle \underline{\cos 110°}, \sin 110° \rangle + R\langle \underline{\cos 60°}, \sin 60° \rangle$$

$$+ \langle \underline{0}, -100 \rangle = \langle 0, 0 \rangle$$

$\Downarrow$ leads

① $L\cos 110° + R\cos 60° = 0$

② $L\sin 110° + R\sin 60° = 100$

① $\Rightarrow \quad L = \dfrac{-R\cos 60°}{\cos 110°}$

② $\Rightarrow \quad \dfrac{-R\cos 60° \sin 110°}{\cos 110°} + R\sin 60° = 100$

$$R = \frac{100}{\sin 60° - \dfrac{\cos 60° \sin 110°}{\cos 110°}} =$$

44.6826 lb

$$L = \frac{-R\cos 60°}{\cos 110} = 65.2704 \text{ lb}$$

§ 11.3 <u>Dot product</u>

<u>Defn</u> If $\bar{u} = \langle u_1, u_2, u_3 \rangle$

and $\bar{v} = \langle v_1, v_2, v_3 \rangle$, the <u>dot product</u> is

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

(similar for $\bar{u}, \bar{v}$ in $\mathbb{R}^2$)

<u>Ex1</u>

(a) $\langle 1, -3, 2 \rangle \cdot \langle 3, 10, 8 \rangle =$

$3 + (-30) + 16 = -11$

(b) $\langle 2, 3, 4 \rangle \cdot \langle 2, 3, 4 \rangle =$

$4 + 9 + 16 = 29$

(c) $\langle 3, 2, -4 \rangle \cdot \langle -1, 1, 1 \rangle = 0$

(d) $\langle 5, 2 \rangle \cdot \langle 8, -4 \rangle =$

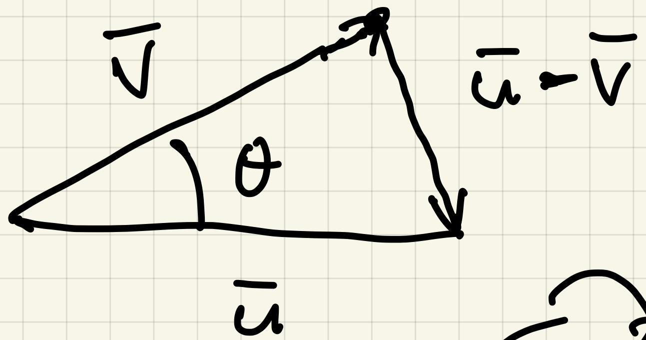
$40 - 8 = 32$

(Easy) <u>Properties</u>

1. $\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$
2. $\bar{u}\cdot(\bar{v}+\bar{w}) = \bar{u}\cdot\bar{v} + \bar{u}\cdot\bar{w}$
3. $c\,(\bar{u}\cdot\bar{v}) = (c\bar{u})\cdot\bar{v} = \bar{u}\cdot(c\bar{v})$
4. $\bar{0}\cdot\bar{v} = 0$
5. $\bar{v}\cdot\bar{v} = |\bar{v}|^2$

<u>Theorem</u> If θ is the angle between vectors $\bar{u}$ and $\bar{v}$,

then $$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|}$$



Law of cosines:

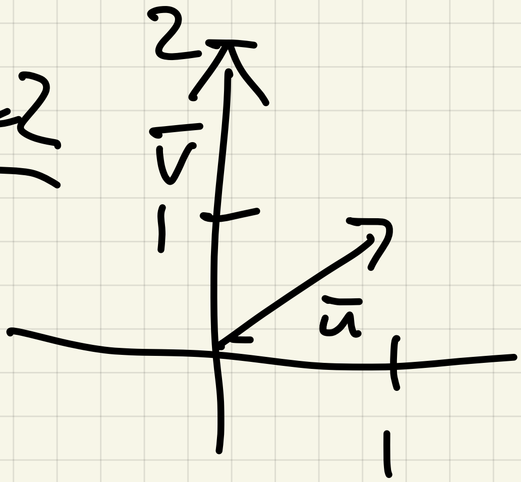
$$|\bar{u}-\bar{v}|^2 = |\bar{u}|^2 + |\bar{v}|^2 - 2|\bar{u}||\bar{v}|\cos\theta$$

$$\cancel{\bar{u}\cdot\bar{u}} - 2\bar{u}\cdot\bar{v} + \cancel{\bar{v}\cdot\bar{v}} = \cancel{\bar{u}\cdot\bar{u}} + \cancel{\bar{v}\cdot\bar{v}} - 2|\bar{u}||\bar{v}|\cos\theta$$

$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos\theta$ ✓

Ex 2

(a)



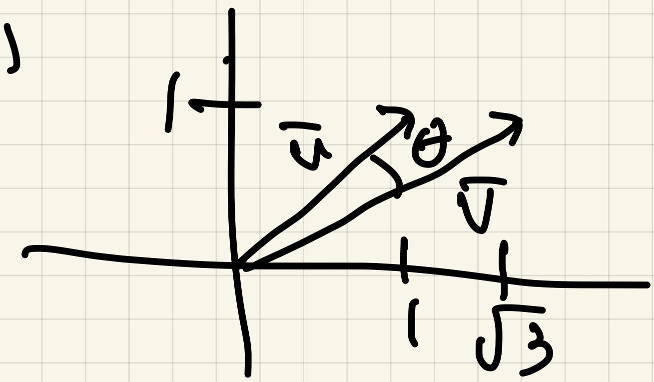
$\bar{u} = \langle 1, 1\rangle$

$\bar{v} = \langle 0, 2\rangle$

$$\frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \frac{2}{\sqrt{2}\cdot 2} = \frac{1}{\sqrt{2}} = \cos\theta$$

so $\theta = 45^\circ$

(b)



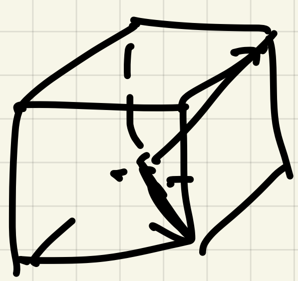
$\bar{u} = \langle 1, 1\rangle$

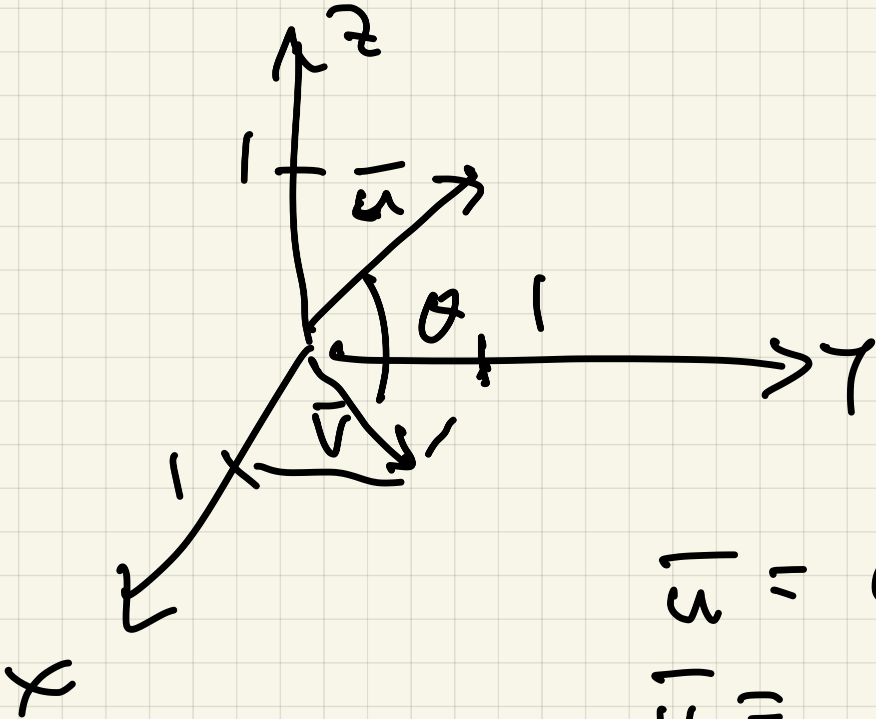
$\bar{v} = \langle \sqrt{3}, 1\rangle$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \frac{\sqrt{3}+1}{\sqrt{2}\cdot 2}$$

$\theta = 15^\circ$

(c)





$\theta = ??$

$\bar{u} = \langle 0, 1, 1 \rangle$

$\bar{v} = \langle 1, 1, 0 \rangle$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

so $\theta = 60^\circ$