4/8 Calc 3

Quiz 15 avg 78

1. $\int_{-3}^{3}\int_{0}^{x+3}\underbrace{\int_{0}^{y} dz}\, dy\, dx$

$\int_0^{x+3} y\, dy = \frac{1}{2}y^2 \Big|_0^{x+3}$

$\int_{x=-3}^{x=3} \frac{1}{2}(x+3)^2 =$ $\boxed{u = x+3}$ $du = dx$

$\shortparallel$

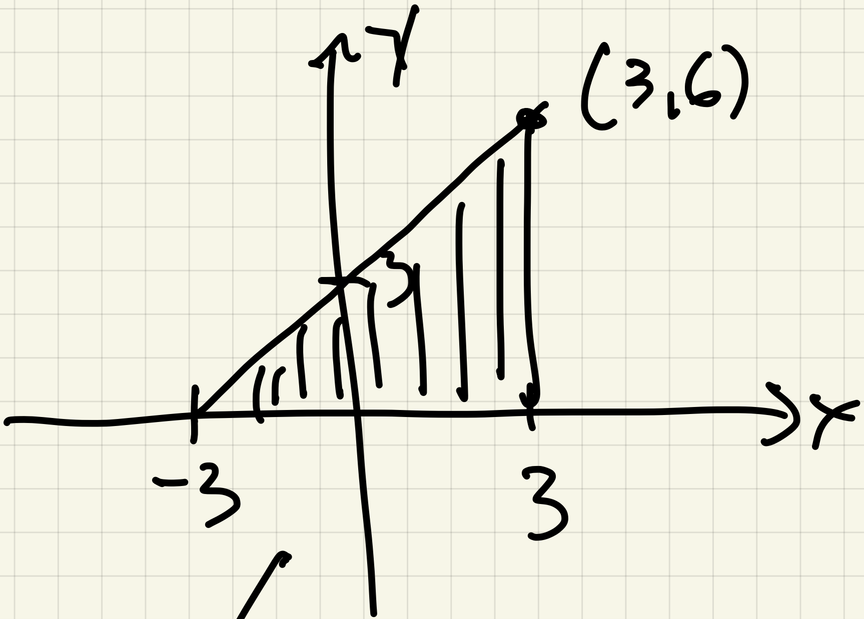
$\int_{u=0}^{u=6} \frac{1}{2}u^2\, du = \frac{1}{6}u^3 \Big|_0^6 =$

$\frac{216}{6} = 36$

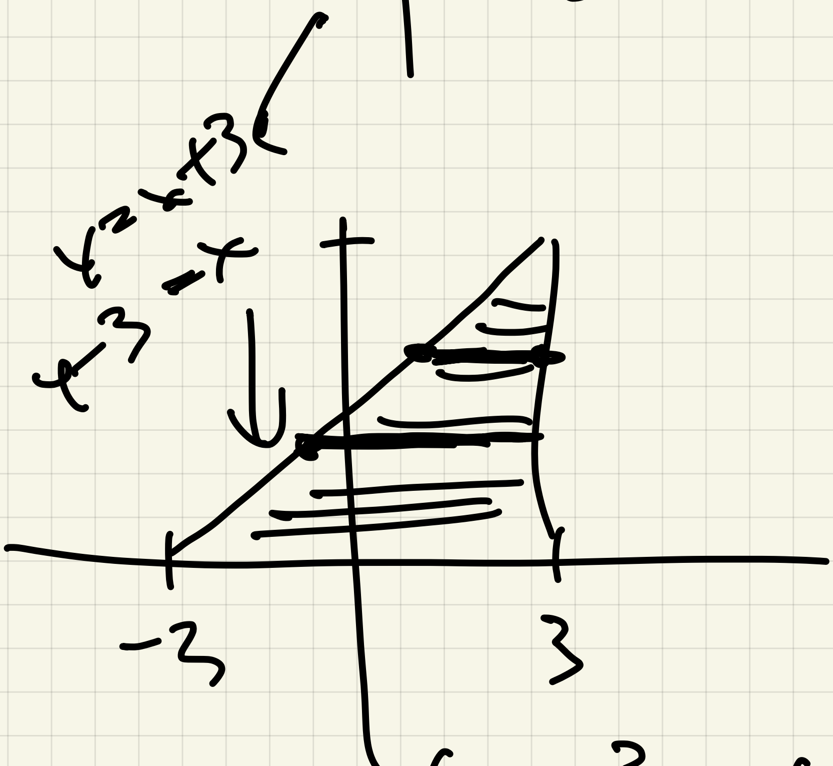
2. $\int_{\boxed{-3}}^{\boxed{3}}\int_{\boxed{0}}^{\boxed{x+3}}$

$-3 \le x \le 3$

$0 \le y \le x+3$



$y = x + 3$



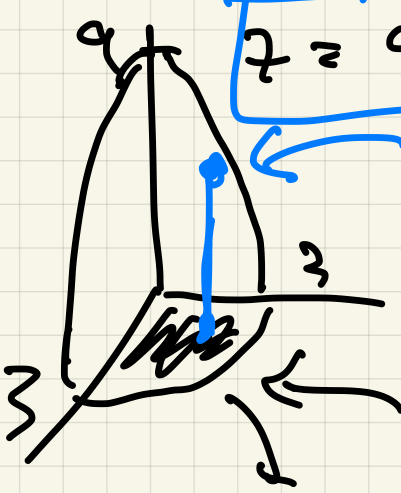
$0 \le y \le 6$

$y-3 \le x \le 3$

$$\int_0^6 \int_{y-3}^{3} \int_0^{y} dz\,dx\,dy$$

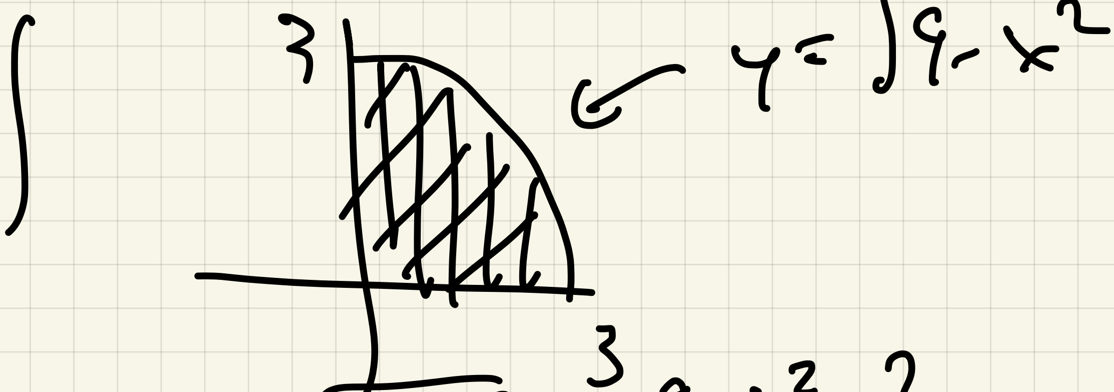
$\boxed{z = 9 - x^2 - y^2}$

$z =$



circle

$x^2 + y^2 = 9$



$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} dz\,dy\,dx$$

Last time: Triple integrals

$$\begin{cases} \text{rectangular} & dx\,dy\,dz & \\ \text{cylindrical} & dz\,dr\,d\theta & r \\ \text{spherical} & d\rho\,d\phi\,d\theta & \rho^2\sin\phi \end{cases}$$

volume

Ex 0 #59 Region below cone $z=\sqrt{x^2+y^2}$ above sphere

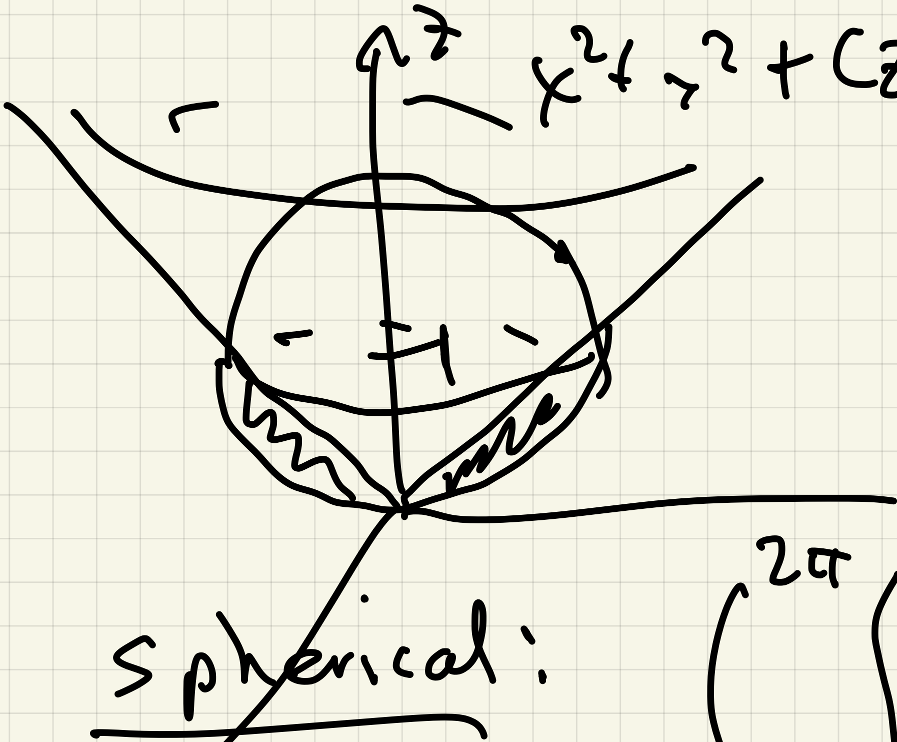
$$\rho = 2\cos\phi$$

$$\underbrace{\rho^2}_{x^2+y^2+z^2} = 2\underbrace{\rho\cos\phi}_{z}$$

$$x^2+y^2+z^2=2z$$

$$x^2+y^2+z^2-2z+1=1$$

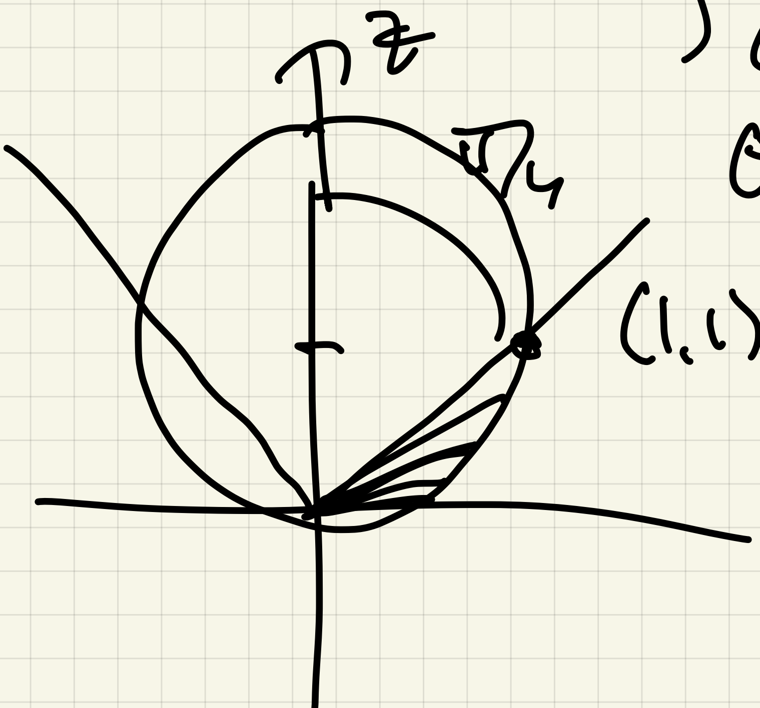
$$x^2+y^2+(z-1)^2=1$$



spherical:

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\underbrace{\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho}\,d\phi\,d\theta$$

θ



$$\left.\frac{\rho^3}{3}\sin\phi\right|_0^{2\cos\phi} =$$

$$\frac{8\cos^3\phi}{3}\sin\phi$$

$$\frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\theta$$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

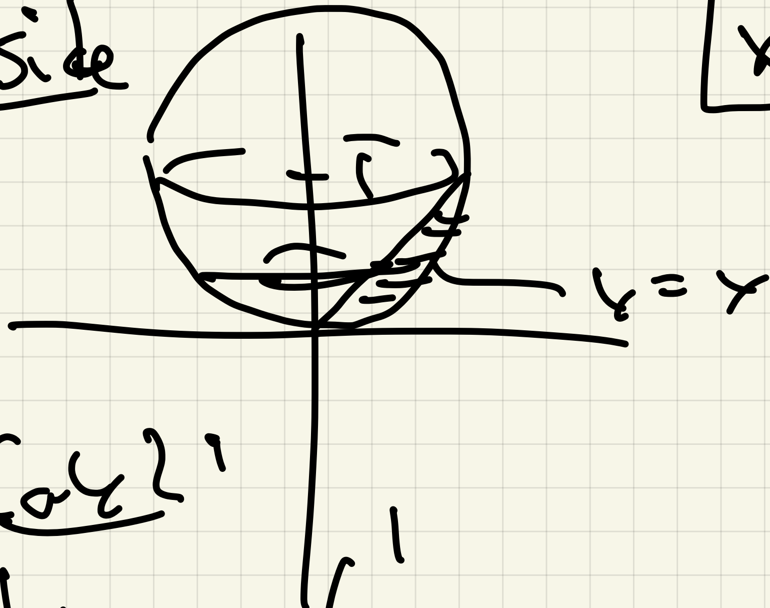
$$\int_{\frac{1}{\sqrt{2}}}^{0} -u^3 \, du = \int_0^{\frac{1}{\sqrt{2}}} u^3 \, du =$$

$$\left.\frac{u^4}{4}\right|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{16}$$

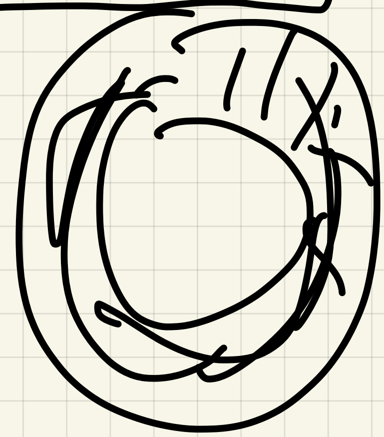
$$\frac{8}{3}\int_0^{2\pi} \frac{1}{16} = \frac{8}{3\cdot 16} 2\pi = \frac{\pi}{3}$$

Alsu =

Aside



$\boxed{x^2+(y-1)^2=1}$



Case 2:

$$V = \int_0^1 \pi\left(\text{out}^2 - \text{in}^2\right)dy$$

$$\int_0^1 \pi\left(\left(\sqrt{1-(y-1)^2}\right)^2 - y^2\right)dy$$

$$\pi\int_0^1 1-(y-1)^2 - y^2 \, dy$$

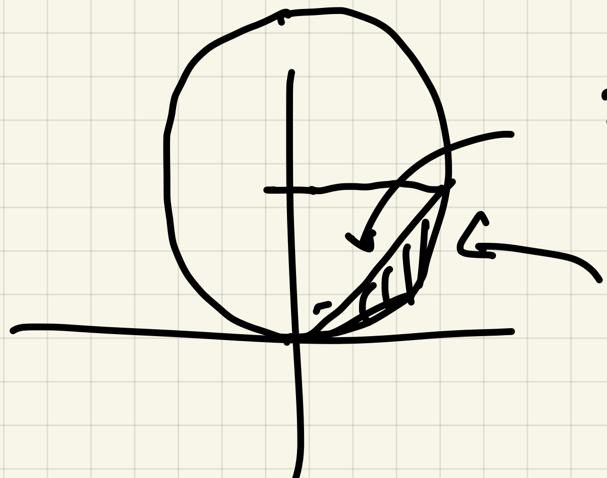
$$\pi \; y - \frac{(y-1)^3}{3} - \frac{y^3}{3}\Big|_0^1$$

$$\pi\left(1 - 0 - \frac{1}{3}\right) - \left(0 + \frac{1}{3} - 0\right)$$

$$= \pi\left(1 - \frac{1}{3} - \frac{1}{3}\right) = \frac{\pi}{3} \checkmark$$

Cylindrical:

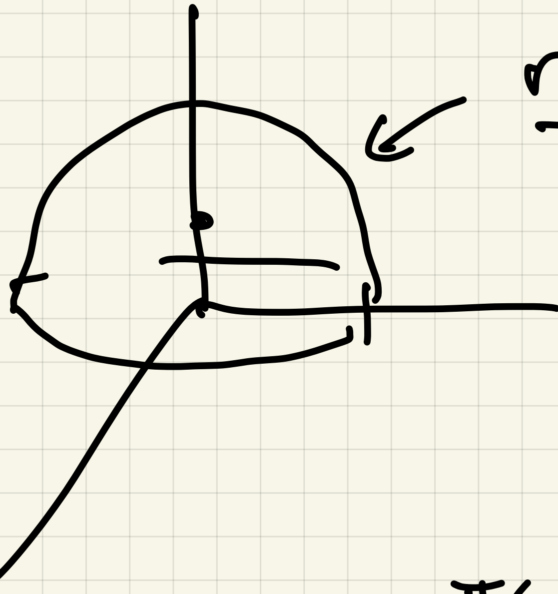
$$\int_0^{2\pi}\int_0^1\int_{1-\sqrt{1-r^2}}^{r} r\,dz\,dr\,d\theta$$



Ex 1 If B = upper solid hemisphere radius 1, compute

$$\bar{z} = \frac{\iiint_B z\,dV}{V} = \text{z-coordinate of center of mass}$$

$$V = \iiint_B dV$$



region:

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$

$0 \le \rho \le 1$

$$\int_0^{2\pi}\int_0^{\pi/2}\underbrace{\int_0^1 \rho^2 \sin\phi \, d\rho}\, d\phi \, d\theta$$

$$\frac{1}{3}\rho^3 \sin\phi \Big|_0^1 =$$

$$\int_0^{\pi/2} \frac{1}{3}\sin\phi \, d\theta =$$

$$-\frac{1}{3}\cos\phi \Big|_0^{\pi/2} = 0-\left(-\frac{1}{3}\right)$$

$$\int_0^{2\pi} \frac{1}{3}\, d\theta = \frac{2\pi}{3} \checkmark$$

Try:
$$\iiint_B \underline{z}\, dV =$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho \underline{\cos\phi}\, \rho^2 \underline{\sin\phi}\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \cos\phi \sin\phi \int_0^1 \rho^3 d\rho$$

$$\frac{\rho^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\int_0^{\pi/2} \frac{1}{4}\cos\phi \sin\phi$$

$u = \sin\phi$

$du = \cos\phi$

$$\frac{1}{8}\sin^2\phi \Big|_0^{\pi/2} = \int_0^{2\pi} \frac{1}{8}\, d\theta =$$

$$\frac{2\pi}{8} = \frac{\pi}{4}$$

$$\xi = \frac{\pi/4}{2\pi/3} = \frac{\cancel{\pi}}{4} \cdot \frac{3}{2\cancel{\pi}} = \frac{3}{8}$$

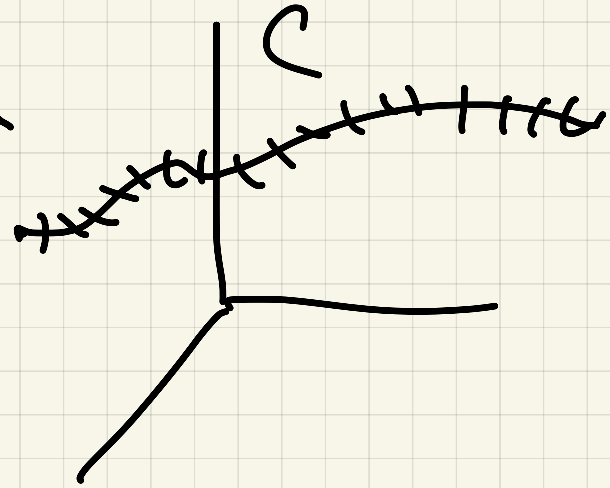
Ch 15

15.1 line integrals of scalar function

Calc $\int_a^b f(x)\,dx$

Calc 3 $\iint_R f(x,y)\,dA$ $\iiint_B f(x,y,z)\,dV$

line integral :



C curve

$f(x,y,z)$ function

Want to integrate f over C

<u>Concept</u> : Break up C into small segments of length Δs_k

$$1 \le k \le n$$

Choose (x_k, y_k, z_k) in k^{th} segment

$$\int_C f(x,y,z)\,ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k) \cdot \Delta s_k$$

<u>How to calculate it??</u>

If C is given by

$\bar{r}(t) \qquad a \le t \le b$, then

$$\int_C f\,ds = \int_{t=a}^{t=b} f(r(t))\,|v(t)|\,dt$$

↑ Notation // ↑ speed

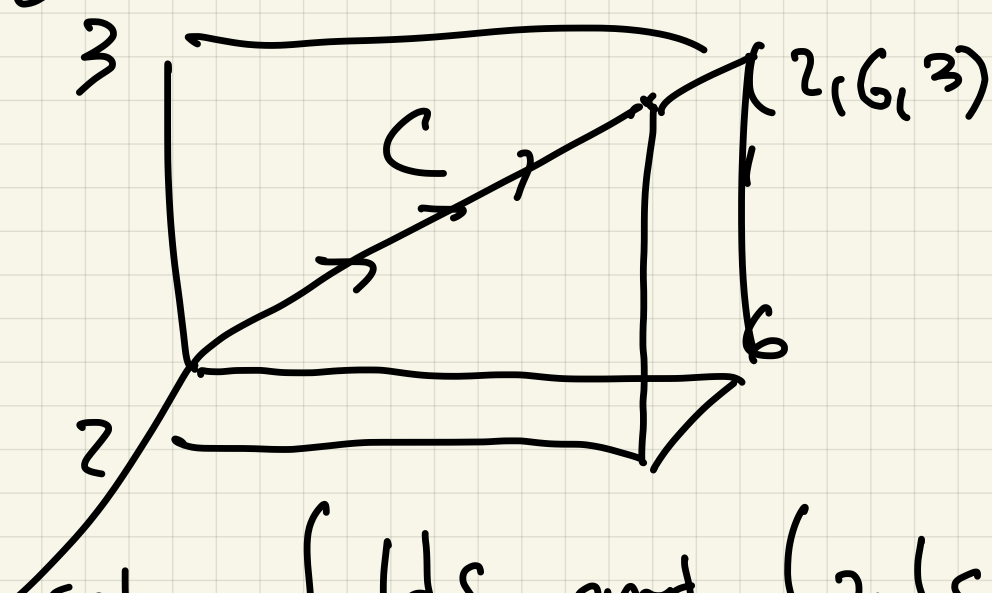
$$\int f(r(t)) \cdot |r'(t)|\,dt$$

Physical interpretation:

① $\int 1\,ds$ = arc length

② If f gives density of curve at position, then
$\int f\,ds$ = mass of curve

Ex1



Find $\int_C 1\,ds$ and $\int_C z\,ds$

Parametrize C:

$$r(t) = (2t, 6t, 3t), \quad 0 \le t \le 1$$

$$v(t) = (2, 6, 3)$$

$$|v(t)| = |(2,6,3)| = \sqrt{2^2 + 6^2 + 3^2} = 7$$

(a)

So $$\int_C 1\, ds = \int_0^1 1 \cdot 7\, dt = 7t \Big|_0^1 = 7$$

(b) $$\int_C \boxed{z}\, ds = \int 3t \quad 7\, dt$$

$$\int_0^1 21t = \frac{21t^2}{2}\Big|_0^1 = \frac{21}{2}$$

BTW ~~cofm~~ Center of mass

z coordinates

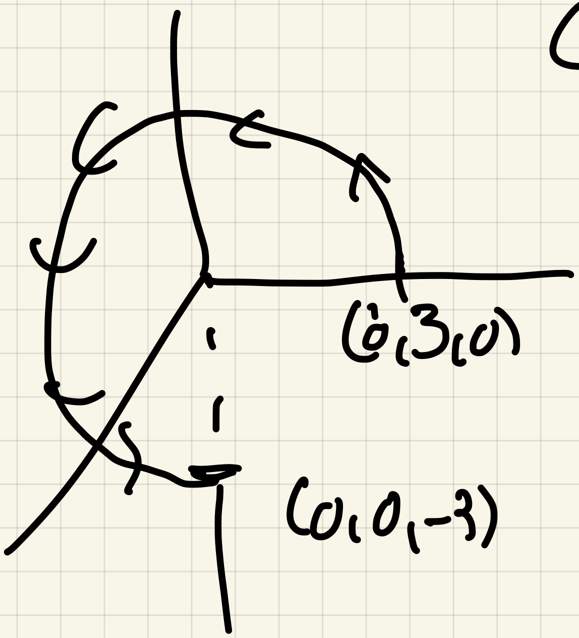
$$\bar{z} = \frac{\int_C z\,ds}{\int_C 1\,ds} = \frac{21/2}{7} = \frac{3}{2} \checkmark$$

midpoint = $(1, 3, 3/2)$

Ex) Compute

$$\int_C z^3 + x\,ds$$

(a)



$$\vec{r}(t) = \langle 0, 3\cos t, 3\sin t \rangle$$

$$0 \le t \le 3\pi/2$$

$$\vec{r}\,'(t) = \langle 0, -3\sin t, +3\cos t \rangle$$

$$|\bar{r}'(t)| = \sqrt{0 + 9\underbrace{\sin^2 t + 9\cos^2 t}}$$

$$= \sqrt{9} = 3$$

Sub

$$\int_C \underline{\underline{z^3 + x}}\, ds = \int_0^{3\pi/2} \left((3\sin t)^3 + 0 \right) \overset{d}{3}\, dt$$

$$\int_0^{\pi/2} 81\, \underline{\underline{\sin^3 t}}\, dt =$$

$$81 \int_0^{\pi/2} (1 - \cos^2 t) \sin t \, dt$$

$$\parallel \qquad \begin{aligned} u &= \cos t \\ du &= -\sin t\, dt \end{aligned}$$

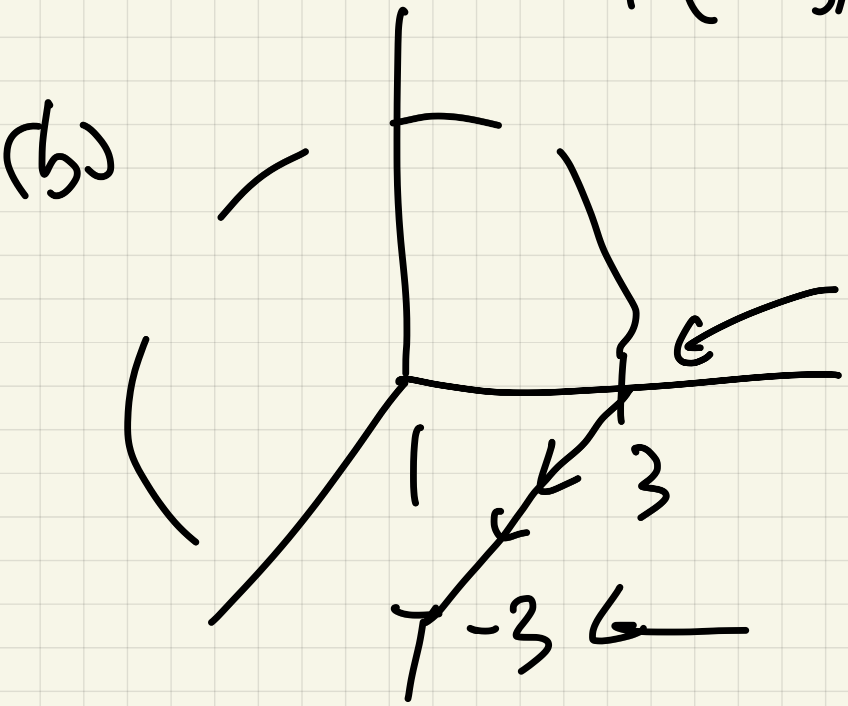
$$81 \int_1^0 -(1 - u^2)\, du =$$

$$81 \int_0^1 (1 - u^2)\, du =$$

$$81 \quad u - \frac{u^3}{3} \Big|_0^1 =$$

$$81\left(\tfrac{2}{3}\right) = 54$$

(b)



$(0,3,0)$

$(0,0,-3)$

direction is $(0,-3,-3)$

<u>parametrize</u>

$$\bar{r}(t) = (0,3,0) + t(0,-3,-3)$$

$$= (0,\ 3-3t,\ -3t)$$

$x \quad y \quad z \qquad 0 \le t \le 1$

$$\bar{r}'(t) = \langle 0,-3,-3 \rangle$$

$$|\bar{r}'(t)| = \sqrt{0+9+9} = 3\sqrt{2}$$

So $\int_C z^3 + \cancel{x}\, ds =$ $\qquad f(r(t))$

$$\int_0^1 \left((-3t)^3 + 0\right) \cdot 3\sqrt{2}\, dt$$

$$\int_0^1 -81\sqrt{2}\, t^3$$

$$-81\sqrt{2}\, \frac{t^4}{4} \Big|_0^1 =$$

$$-\frac{81\sqrt{2}}{4}$$