4/15 | Calc3

Vector fields

Last time

$$\int_C F \cdot dr = \int M\,dx + N\,dy + P\,dz$$

$C \leftarrow$ oriented curve

A vector field F is conservative

if $F = \nabla f \leftarrow$ f = potential function

① $F(x,y) = (M, N)$ conservative

$\Updownarrow$

$$\partial M/\partial y = \partial N/\partial x \leftarrow$$

② $F(x,y,z) = (M, N, P)$ conservative

$\Updownarrow$

$$\partial M/\partial y = \partial N/\partial x$$

$$\partial M/\partial z = \partial P/\partial x$$

$$\partial N/\partial z = \partial P/\partial y$$

$\nabla f = (f_x, f_y, f_z)$

Ex 1

$$F(x,y) = (\underbrace{x^3+y+\sin y}_{M}, \underbrace{y+x+x\cos y}_{N})$$

$$M_y = 1+\cos y \overset{\checkmark}{=} N_x = 1+\cos y$$

$f = ??$ $f_x = x^3+y+\sin y$

$$f = \int x^3+\underline{y}+\sin y \, dx$$

function of y

$$= \frac{1}{4}x^4 + xy + x\sin y + \underline{C(y)}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{4}x^4 + xy + x\sin y + C(y)\right)$$

$$= 0 + \underline{x + x\cos y} + C'(y)$$

$$= y + \underline{x + x\cos y}$$

Take $C(y) = \frac{1}{2}y^2$

5, $f(x,y) = \frac{1}{4}x^4 + xy + x\sin xy + \frac{1}{2}y^2$

(b) $F(x,y) = (\underset{M}{2x+5y}, \underset{N}{3x-7y})$

$$M_y = 5 \neq N_x = 3$$

no potential function.

<u>not</u> conservative

(c) $F(x,y,z) = \left(\underset{M}{-\frac{z}{x^2}}, \underset{N}{\frac{1}{z}}, \underset{P}{-\frac{y}{z^2} + \frac{1}{x}}\right)$

$M_y = N_x = 0$

x^{-1}

$M_z = -\frac{1}{x^2}$, $P_x = -\frac{1}{x^2}$

$\partial N/\partial z = -\frac{1}{z^2} = \partial P/\partial y = -\frac{1}{z^2}$

IS conservative ;

$$f_x = -\frac{z}{x^2} \qquad f_y = \frac{1}{z}$$

$$f_z = -\frac{y}{z^2} + \frac{1}{x}$$

$$f = \frac{z}{x} + C(y,z)$$

$$\partial f/\partial y = 0 + \partial/\partial y\, C(y,z) = \frac{1}{z}$$

$$C(y,z) = \frac{y}{z}$$

$$f = \frac{z}{x} + \frac{y}{z} + C(z)$$

$$f_z = \frac{1}{x} - \frac{y}{z^2} + \cancel{C'(z)}$$

<u>Ex 2</u> $F(x,y,z) = \langle \underbrace{2xyz}_{M}, \underbrace{x^2 z}_{N}, \underbrace{x^2 y + 3z^2}_{P} \rangle$

$M_y = 2xz$ ✓ $M_z = 2xy$ ✓

$N_x = 2xz$ $P_x = 2xy$

$N_z = x^2$ ✓

$P_y = x^2$

Find f: $f_z = x^2y + 3z^2$

$f = x^2yz + z^3 + C(x,y)$

So check with

M, N

$f_x = M, \; f_y = N$ ✓

$f = x^2yz + z^3$

Fundamental theorem of line integrals (FTLI)

Thm: If $F = \nabla f$ is a

conservative vector field,

and $C : \bar{r}(t),\ a \le t \le b$

Then

$$\int F \cdot dr = \underline{f(r(b))} - f(r(a))$$

chain rule

<u>Why?</u> $\frac{d}{dt}\left(f(r(t))\right) =$

$\quad r(t) = (x(t), y(t), z(t))$

$$\underline{\partial f/\partial x} \cdot \underline{\frac{dx}{dt}} + \underline{\partial f/\partial y} \cdot \underline{\frac{dy}{dt}} + \underline{\partial f/\partial z} \cdot \underline{dz/dt}$$

$$\underbrace{\nabla f}_{F} \cdot r'(t) \Leftarrow$$ is what you would integrate!

<u>Ex 4</u>

$$\int_C \underset{M}{\underline{2xyz\,dx}} + \underset{N}{x^2 z\,dy} + \underset{P}{(x^2 y + 3z^2)\,dz}$$

$C: r(t) = (t^3+1, \cos \pi t, \sin \pi t)$

$0 \le t \le 1$

$$\int_0^1 2(t^3+1)\cos \pi t \sin \pi (3t^2) + \cdots$$

Instead use FTLI

$$F(x,y,z) = (\underset{M}{2xyz}, \underset{N}{x^2z}, \underset{P}{x^2y+3z^2})$$

Is conservative:

$$f = \underline{x^2yz + z^3}$$

$$\int_C F \cdot dr = f(r(1)) - f(r(0))$$

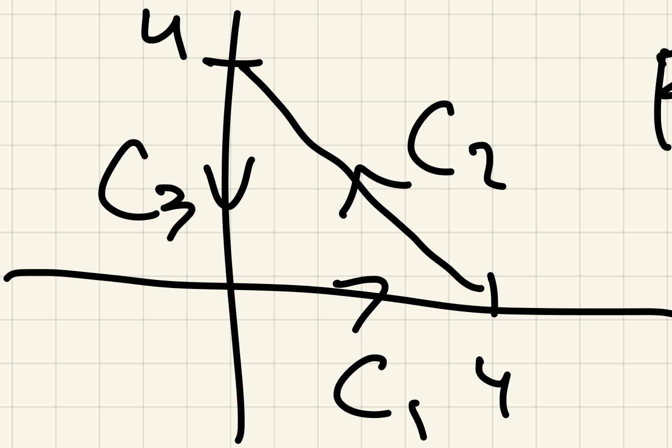
$r(1) = (2, -1, 0)$

$r(0) = (1, 1, 0)$

$f(2, -1, 0) - f(1, 1, 0) = 0$

O - d

Ex 5



$$F(x,y) = (\overset{M}{x}, \overset{N}{y})$$

$$M_y = 0$$

$$N_x = 0$$

$$f = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$r(t)$

$r(0) = (0,0)$

$r(1) = (0,0)$, 0

$$\int_C F \cdot dr = f(0,0) - f(0,0) = 0$$

Note: If C is a closed curve, $r(b) = r(a)$, and

~~then~~ F conservative,

Then $\int_C F \cdot dr = 0$

Theorem: Let F is a vector field

The following are equivalent:

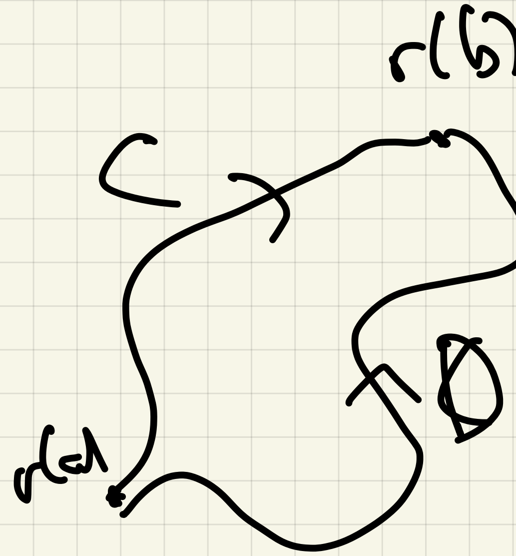
1) F is conservative

2) If C is any closed curve, then $\int_C F \cdot dr = 0$

3) If C, D are two curves with $r(t)$, $s(t)$
$r(a) = s(a)$
$r(b) = s(b)$

then

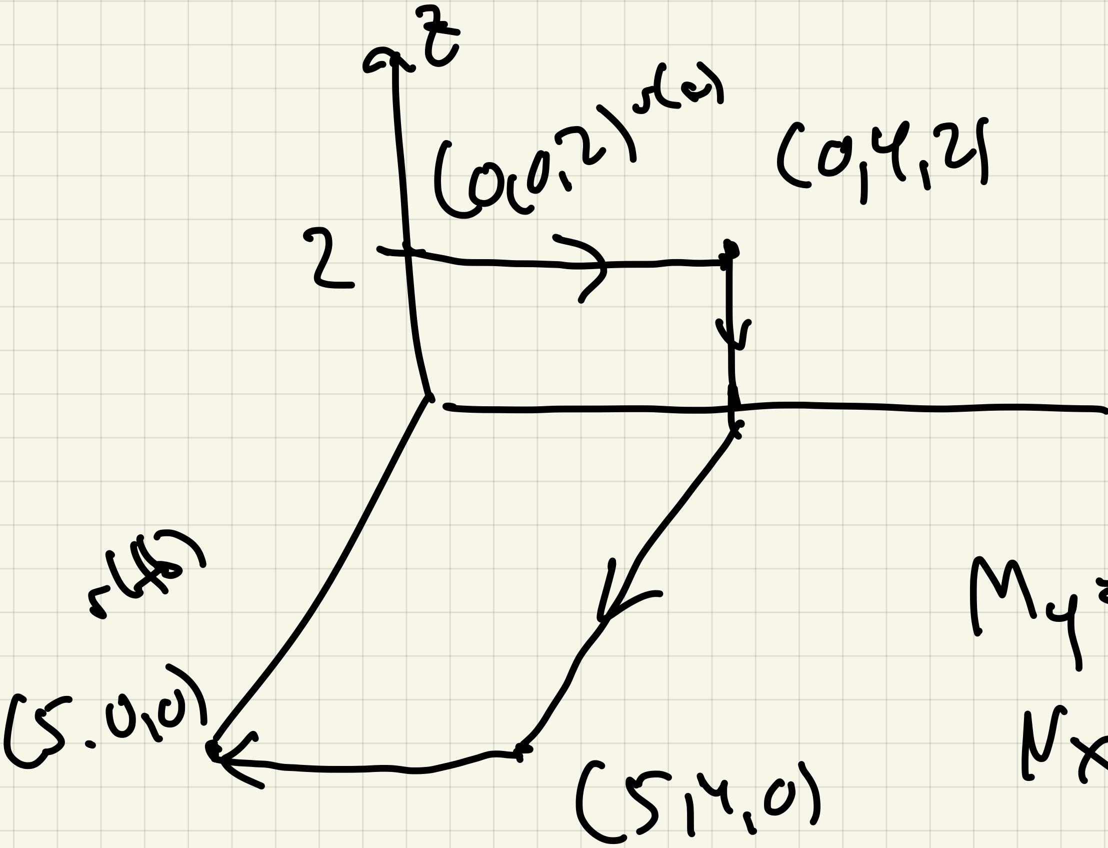
$$\int_C F \cdot dr = \int_D F \cdot dr$$



Path Independence

Ex 6 $\int_C \underbrace{(x+z^2+3y)}_{}^{M}\,dx + \underbrace{(3x-\sin y+z)}^{N}\,dy + \underbrace{(2xz+y+z^3)}_{P}\,dz$

where C is the curve:



Looks unpleasant!

$M_y = 3$ ✓
$N_x = 3$

$M_z = 2z$ ✓ $N_z = 1$
$P_x = 2z$ $P_y = 1$

$(\underbrace{x+z^2}_{}\overset{M}{} + 3y,\ 3x - \sin y + z,\ 2xz + y + z^3)$

$f_x \qquad f_y \qquad f_z \quad \uparrow$

$$f = \int M\,dx = \frac{1}{2}x^2 + xz^2 + 3xy + C(y,z)$$

$$f_y = 3x + \partial C/\partial y = 3x - \sin y + z$$

so $C(y,z) = \cos y + yz$

so $f = 3x + \cos y + yz + xz^2$

$$\partial f/\partial z = y + 2xz$$

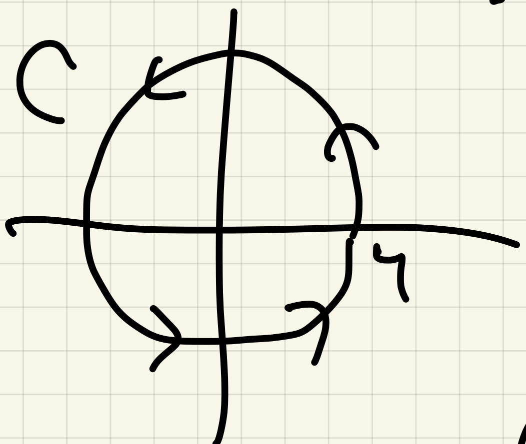
$$C = \frac{1}{4}z^4$$

so $f = 3x + \cos y + yz + xz^2 + \frac{1}{4}z^4$

so $\int_C F \cdot dr = \cancel{A}$

$$f(5,0,0) - f(0,0,2) = \frac{33}{2}$$

Ex 7 $$\int_C \underbrace{(x+3y)}_{M}\,dx + \underbrace{(3x - \sin y)}_{N}\,dy$$



$M_y = 3 = N_x$ ✓

Thm

(M, N) conservative $\Rightarrow$

$$\int_C M\,dx + N\,dy = 0$$

$$\int_C \underbrace{(x+3y)}_{M}\,dx + (\underbrace{2x}_{N} - \sin y)\,dy$$

Not conservative:

$$\int_C 0\,dx + \boxed{x}\,dy + \quad = 0$$

$C : r(t) = (4\cos t, \boxed{4\sin t})$

$0 \le t \le 2\pi$

$$\int_0^{2\pi} 0 + \underline{4\cos t}\cdot\underline{4\cos t}\,dt$$

$$= \int_0^{2\pi} 16\,\underline{\underline{\cos^2 t}}\,dt$$

// double angle

$$\int_0^{2\pi} 16\left(\frac{1+\cos 2t}{2}\right)$$

$$8\int_0^{2\pi} \underline{1+\cos 2t}$$

$$8\left(t+\cancelto{0}{\tfrac{1}{2}\sin 2t}\right)\Big|_0^{2\pi}$$

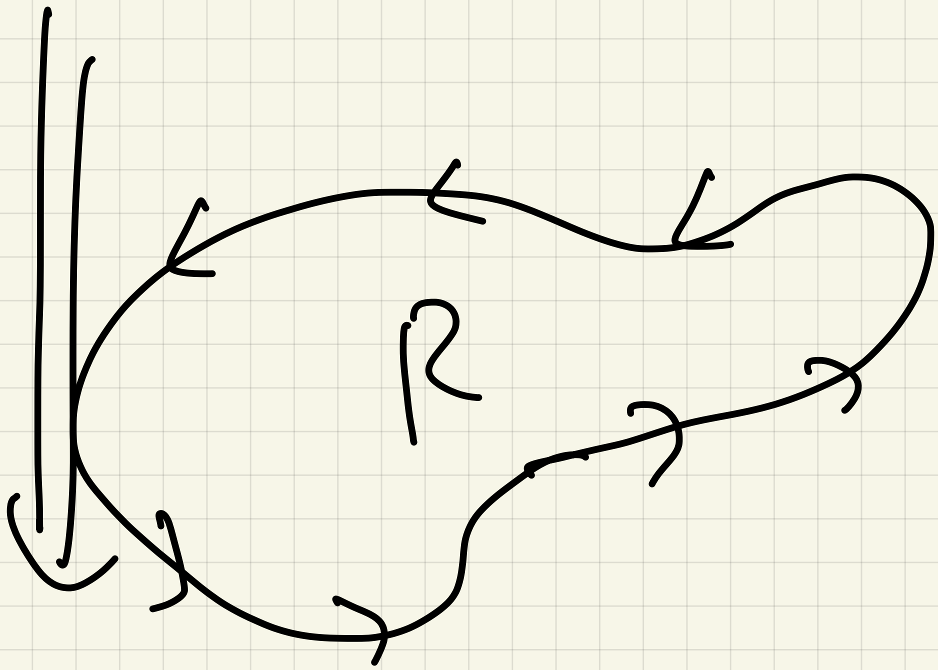
$$8(2\pi) = 16\pi$$

So $\int_C (x+3y)\,dx + (2x-\sin y)\,dy = \cancel{0}\ -16\pi$

<u>Easier method:</u>

Green's Theorem:

If C is a counterclockwise simple closed and bounds a region R,



Then $\int_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$

$$\int_C \overset{M}{(x+3y)}\,dx + \overset{N}{(2x - \sin y)}\,dy$$

$$\parallel$$

$$\iint_R N_x - M_y = \iint_R (2-3)\, dA$$

$$\iint_R -1\, dA = -\text{Area } R$$

$$= -16\pi \checkmark$$