4/10/ Calc3

Last time line integrals of scalar function

Notation $\int_C f\,ds$

Evaluation: parametrize C

$r(t)$ $a \le t \le b$, then

$$\int_C f\,ds = \int_a^b f(r(t))\,|r'(t)|\,dt$$

Note $\int_C 1\,ds =$ arc length

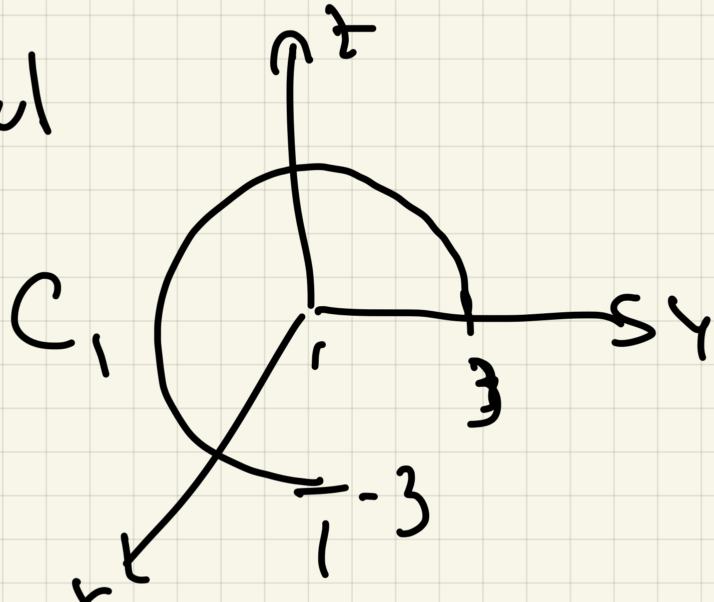
$$\int_C \rho\,ds = \text{mass}$$

ρ = density

Ex 1 Find $\int_C (z^3 + x)\,ds$

for curve C below:

(a)

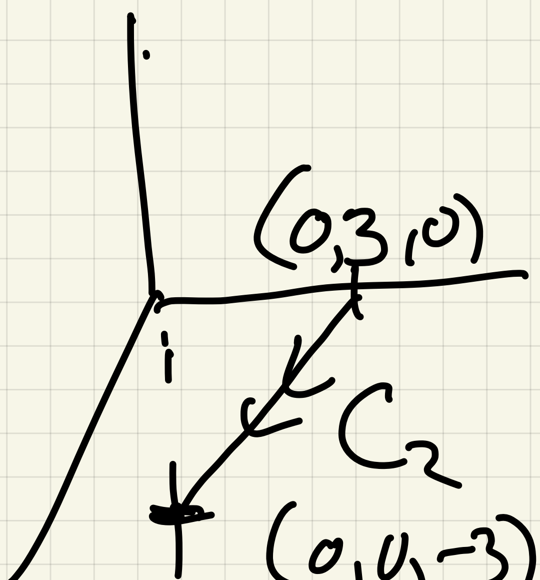


$r(t) = (0, 3\cos t, 3\sin t)$

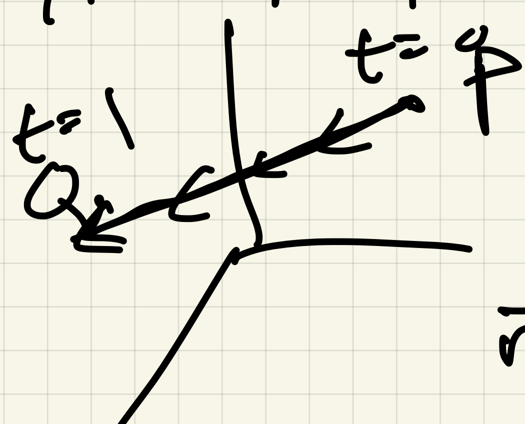
$0 \le t \le 3\pi/2$

$$\int_{C_1} (z^3 + x)\, ds = 54$$

(b)



Recall to parametrize a line segment from P to Q



$v = \overrightarrow{PQ} = \bar{Q} - \bar{P}$

direction

$\bar{r}(t) = P + \underline{\underline{vt}},\ 0 \le t \le 1$

$t=0 \rightarrow P$

$t=1 \Longrightarrow P+(Q-P)=Q$

$C_2: \quad P=(0,3,0)$

$Q=(0,0,-3)$

$v = Q-P = (0,-3,-3)$

so $\quad r(t)= P+vt =$

$(0\ 3\ 0)+t(0,-3,-3)$

$= (0, 3-3t, -3t)$

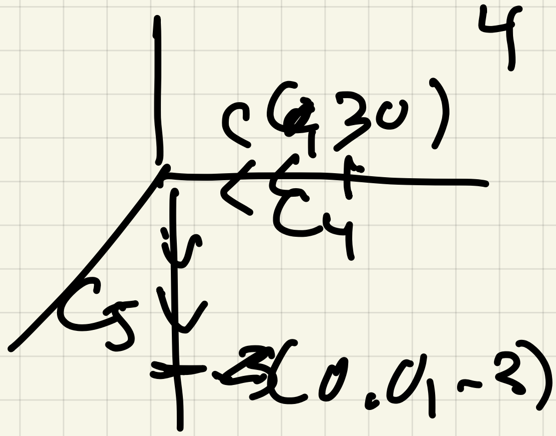
$0 \le t \le 1$

$$\int_{C_2} f\,ds = \int_0^1 \left((-3t)^3+0\right) 3\sqrt{2}\,dt$$

$z^3+x \qquad \Vert$

$$-\frac{81\sqrt{2}}{4}$$

(c) $\quad C_3$



$$C_3 = C_4 + C_5$$

$$\int_{C_3} f\,ds = \underbrace{\int_{C_4} f\,ds}_{①} + \underbrace{\int_{C_5} f\,ds}_{②}$$

① C_4 :

$$r(t) = (0, 3-3t, 0) \qquad 0 \le t \le 1$$

$$\quad x \quad y \quad t$$

$$\int_{C_4} (z^3 + x)\,ds = \int_{C_4} (0^3 + 0)\,ds = 0$$

② C_5 : $r(t) = (0, 0, -3t)$

$$0 \le t \le 1$$

$$\int_{C_5} (z^3 + x)\,dt = \int_0^1 ((-3t)^3 + 0) \cdot \underline{3}\,dt$$

$$r' = (0, 0, -3)$$

$$|r'| = 3$$

$$\int_0^1 -81t^3 =$$

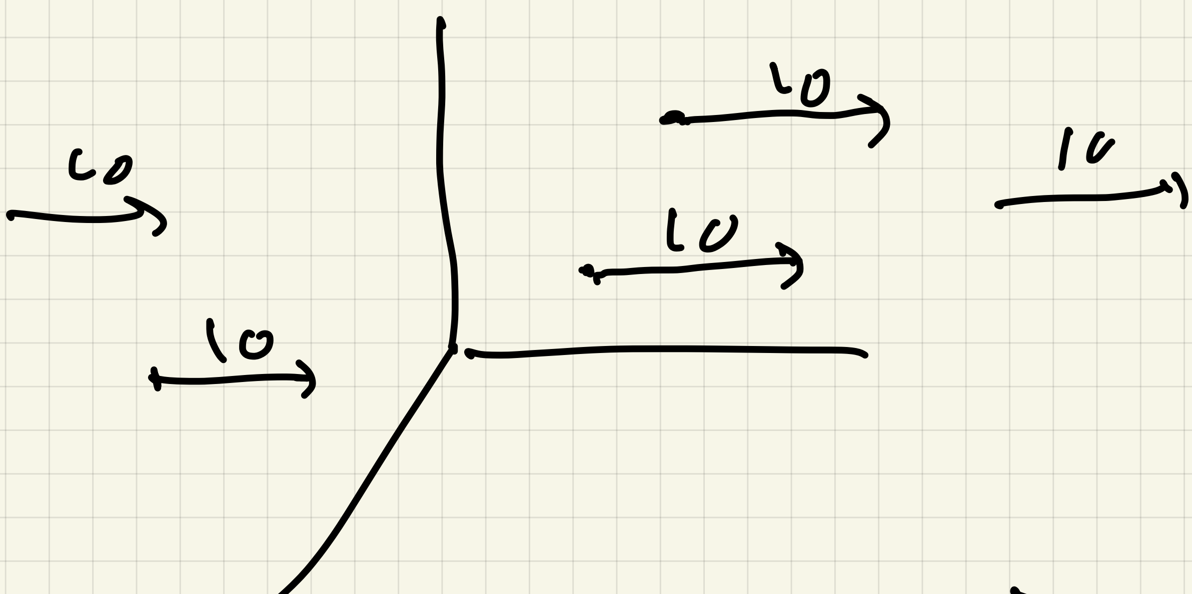
$$-\frac{81t^4}{4}\Big|_0^1 = -\frac{81}{4}$$

§15.2 line integrals over vector fields

A vector field on $\mathbb{R}^3$ is a function : $\mathbb{R}^3 \to \mathbb{R}^3$

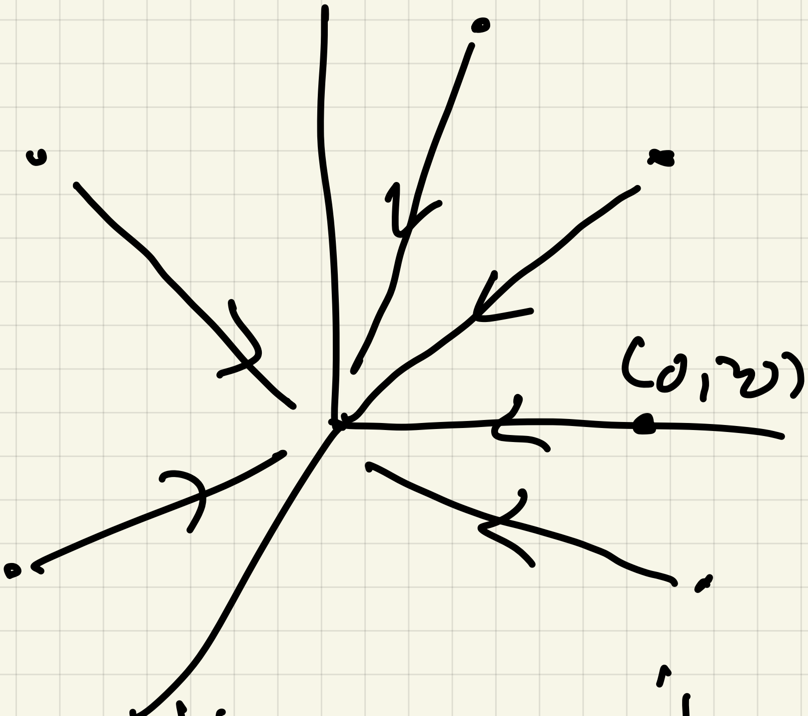
$$F(pt) = \text{vector}$$

Ex1 $F(x,y,z) = \langle 0, 10, 0 \rangle$



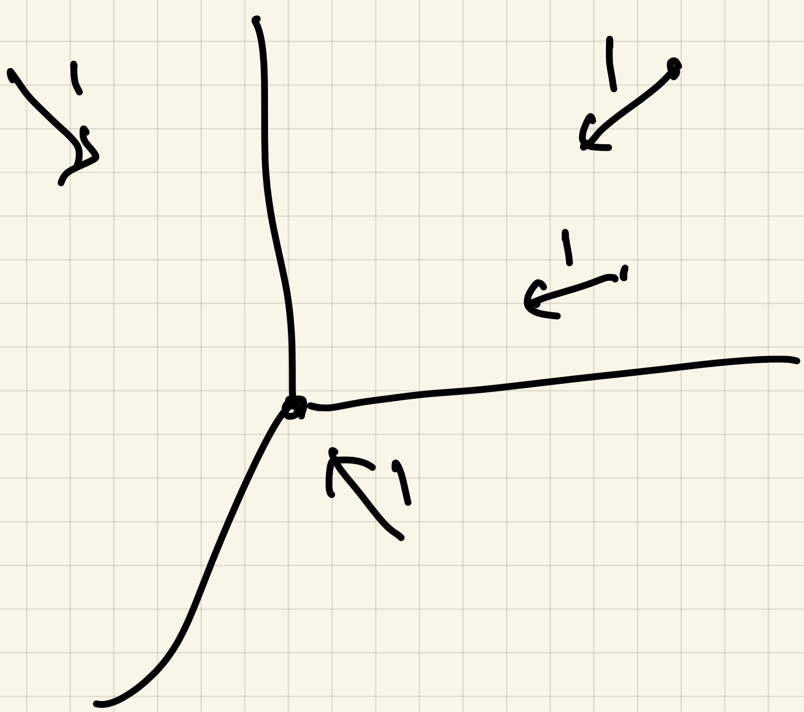
Ex2 $F(x,y,z) = -\langle x,y,z \rangle$

(a)



looks like a gravity field,
but magnitude

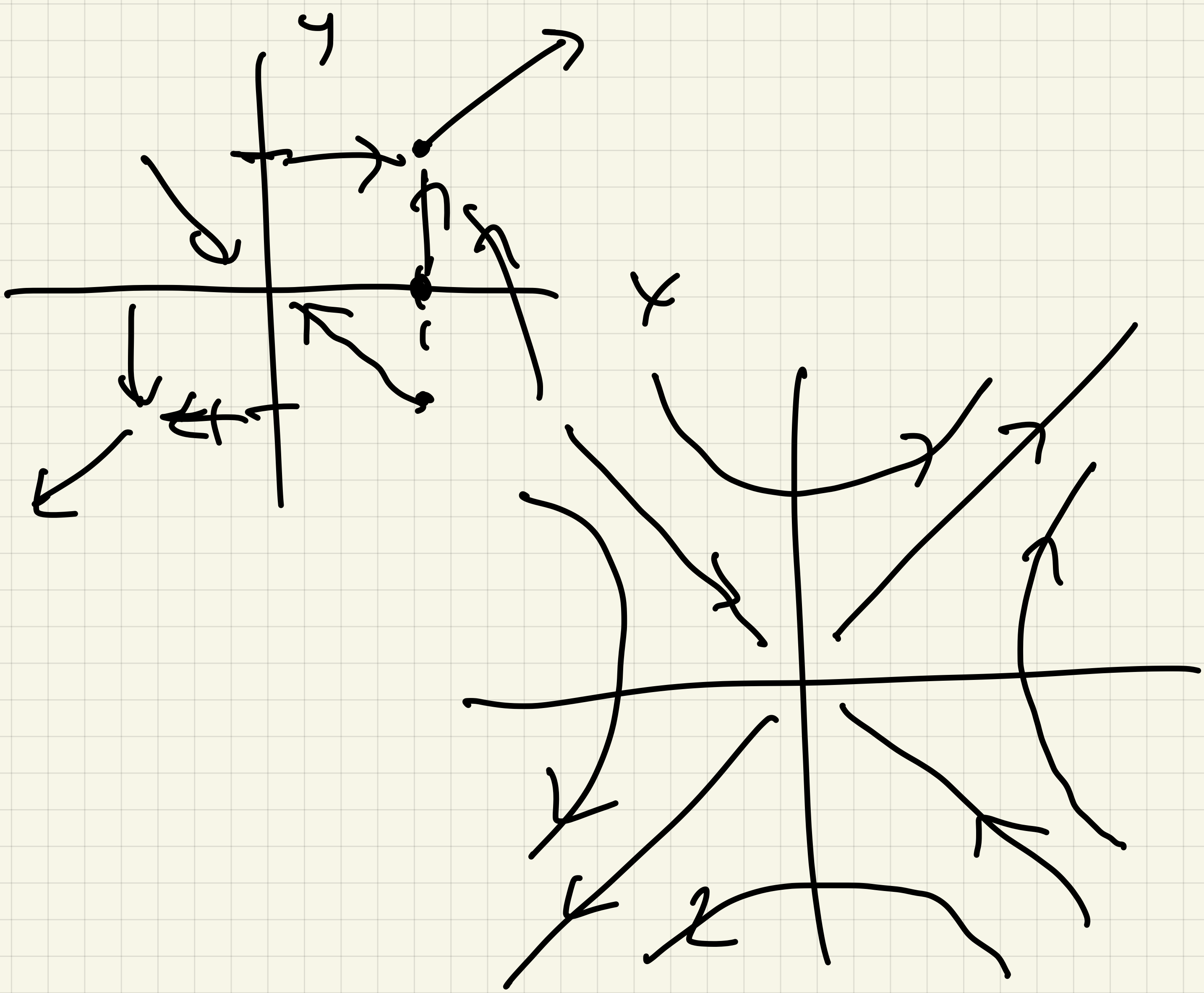
(b) $$F(x,y,z) = -\frac{\langle x,y,z\rangle}{|\langle x,y,z\rangle|}$$



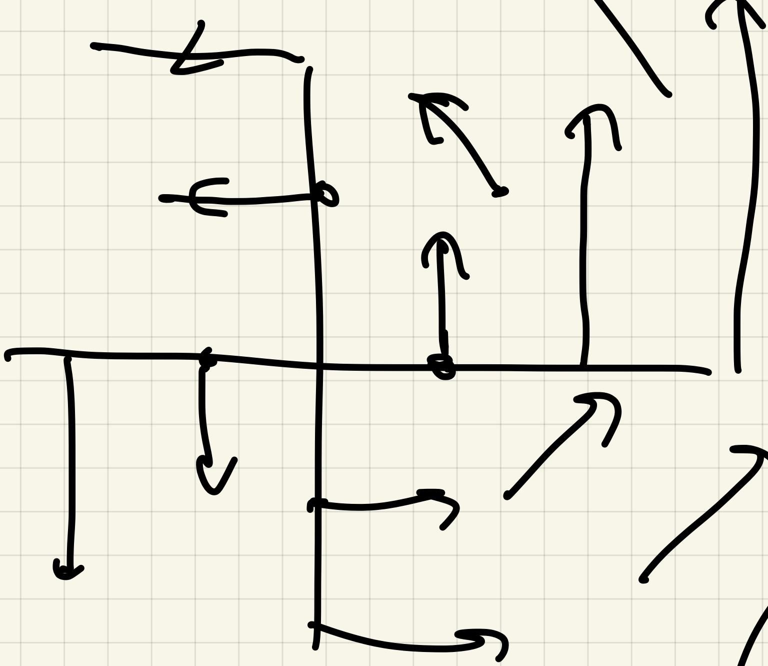
(c) Gravity field:

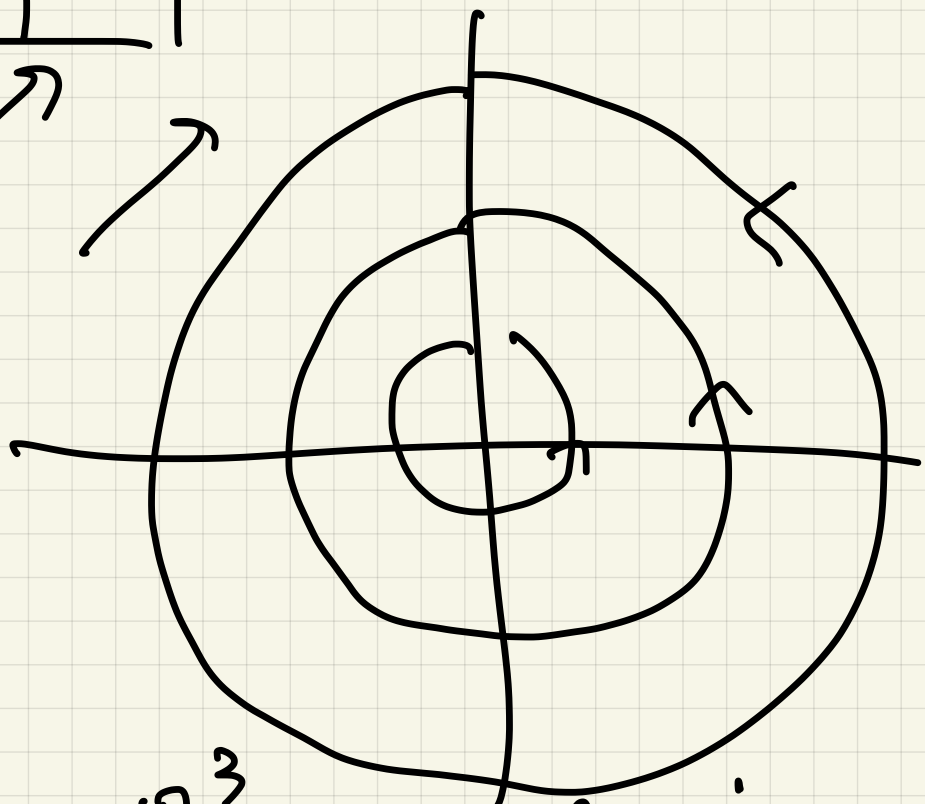
$$G(x,y,z) = \frac{-\langle x,y,z\rangle}{|\langle x,y,z\rangle|^3}$$

Ex 3 $\quad F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad F(x,y) = \langle y, x\rangle$



Ex 4 $\quad F(x,y) = \langle -y, x\rangle$



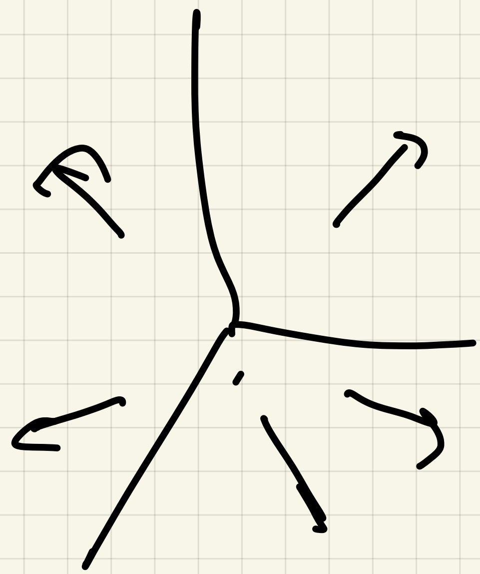


Ex 5 $f: \mathbb{R}^3 \to \mathbb{R}^3$ any function,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$



Now let $C: r(t) =$ oriented curve in $\mathbb{R}^3$
$a \le t \le b$

(A) The line integral of vector field F along C is

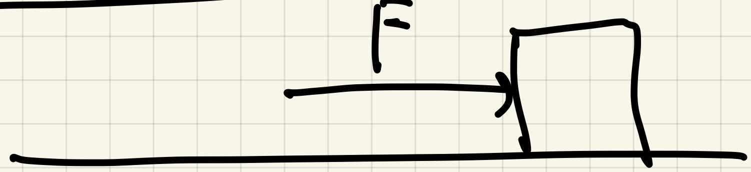
$$\int_C F\cdot T\,ds \;=\; \int_C F\cdot dr$$

vect field — unit tangent — arc length param

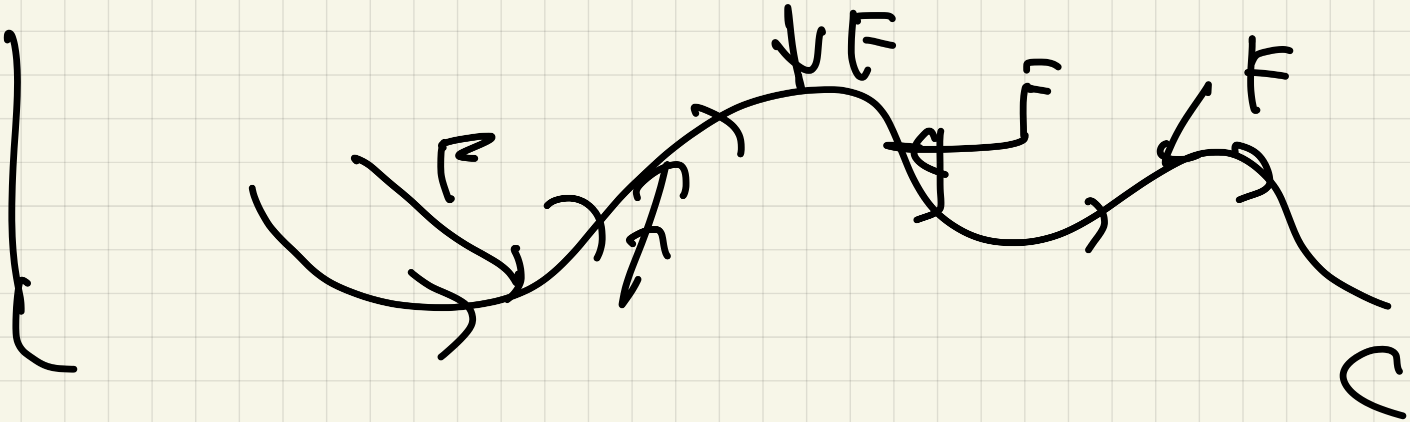
Flow integral
=
Work integral

$$\boxed{\int_a^b F(r(t))\cdot r'(t)\,dt}$$

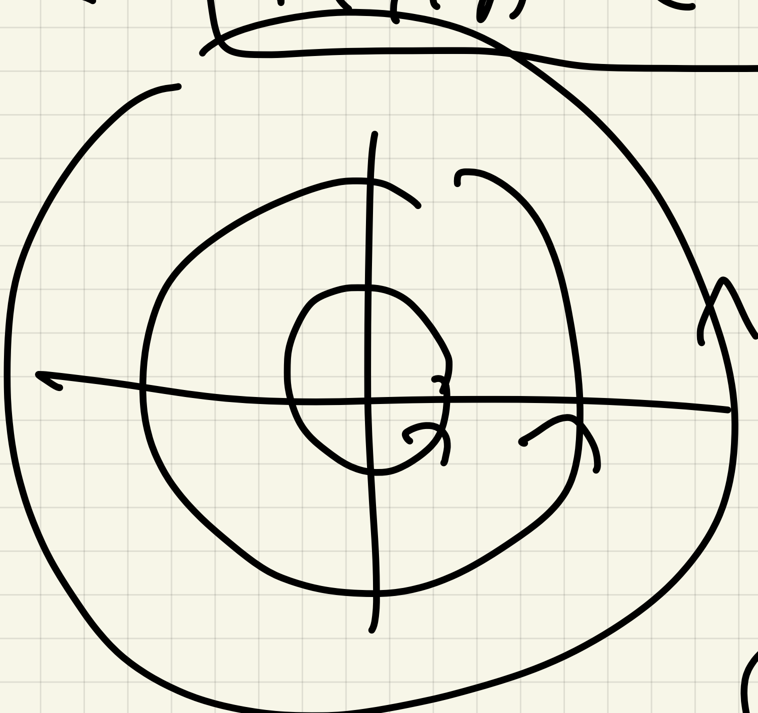
Aside:



Work = Force · distance



Ex 6 $\boxed{F(x,y) = \langle -y, x \rangle}$



Compute $\int_C F \cdot dr$ over curves:

(a) C_1 : $\boxed{\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle}$, $0 \le t \le 2\pi$

$$\int_{C_1} F \cdot dr = \int_0^{2\pi} \langle \underset{-y}{-\sin t}, \underset{x}{\cos t} \rangle \cdot$$

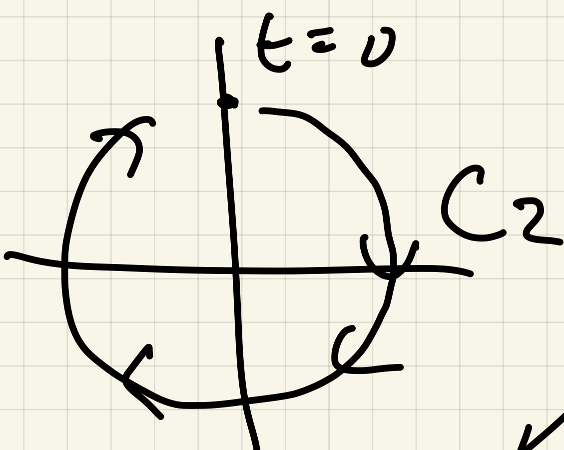
$\langle -\sin t, \cos t \rangle \, dt$

$r'(t)$

$$\int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1} \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$$

(b) C_2 $\quad r(t) = \langle 3\sin t, 3\cos t \rangle$

$0 \le \overset{t}{\theta} \le 2\pi$



$F = \langle -y, x \rangle$

$$\int_{C_2} F \cdot dr = \int_0^{2\pi} \langle -3\cos t, 3\sin t \rangle \cdot \langle 3\cos t, -3\sin t \rangle \, dt$$

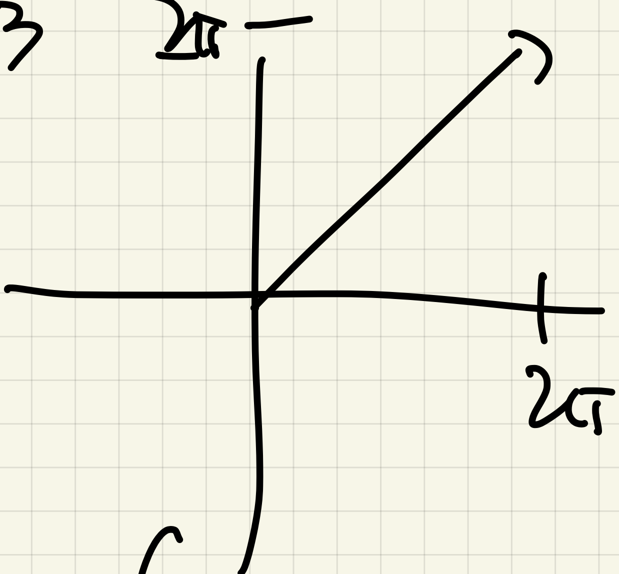
$r' = \langle 3\cos t, -3\sin t \rangle$

$$\int_0^{2\pi} -9\cos^2 t - 9\sin^2 t \, dt = \int_0^{2\pi} -9t$$

$$= -18\pi$$

(c) $r(t) = \langle t, t \rangle, \quad 0 \le t \le 2\pi$

C_3



$r'(t) = \langle 1, 1 \rangle$

$$\int_{C_3} F \cdot dr = \int_0^{2\pi} \langle -t, t \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int_0^{2\pi} 0 \, dt = 0$$

<u>Alternate Notation</u>:

Write $F = (\underset{x}{M}, \underset{y}{N}, \underset{z}{P})$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C M\,dx + N\,dy + P\,dz$$

Ex 7 In Ex 6 $\int_C -y\,dx + x\,dy$

$$= \quad F = \langle -y, x \rangle$$

$$\int_{t=0}^{2\pi} \left(-y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

Ex 8 $\int_C y\,dx$

$C = C_3 \quad r(t) = \langle t, t \rangle, \quad 0 \le t \le 2\pi$

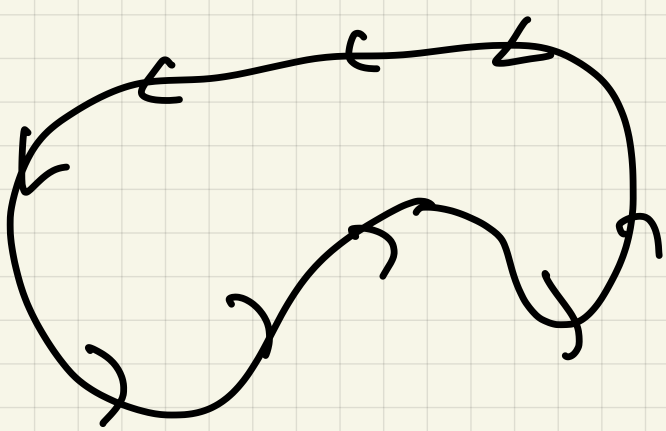
$r(t) = (x(t), y(t))$

$r'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = \langle 1, 1 \rangle$

$$= \int_0^{2\pi} t \cdot 1\,dt =$$

$$\frac{t^2}{2}\Big|_0^{2\pi} = \frac{4\pi^2}{2} = 2\pi^2$$

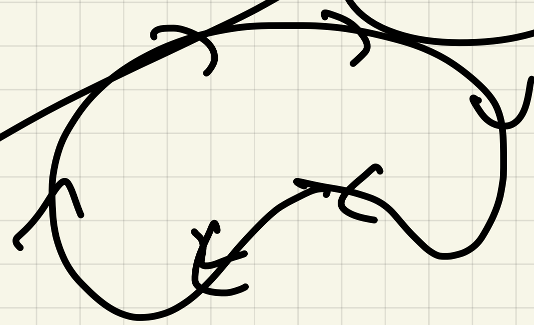
Aside: If C is a closed curve (i.e. $r(a) = r(b)$)



$$\int_C F \cdot dr$$

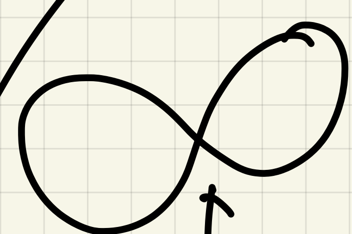
circulation integral

(B) If C is a simple closed curve

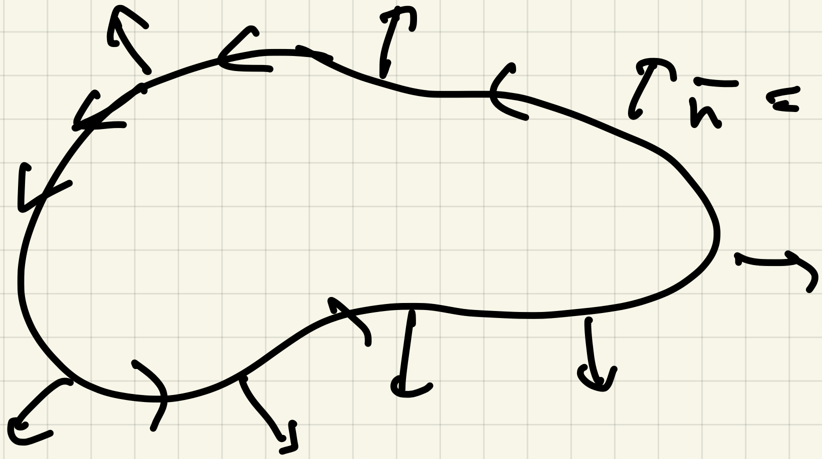


no self-crossing

start = end



not simple

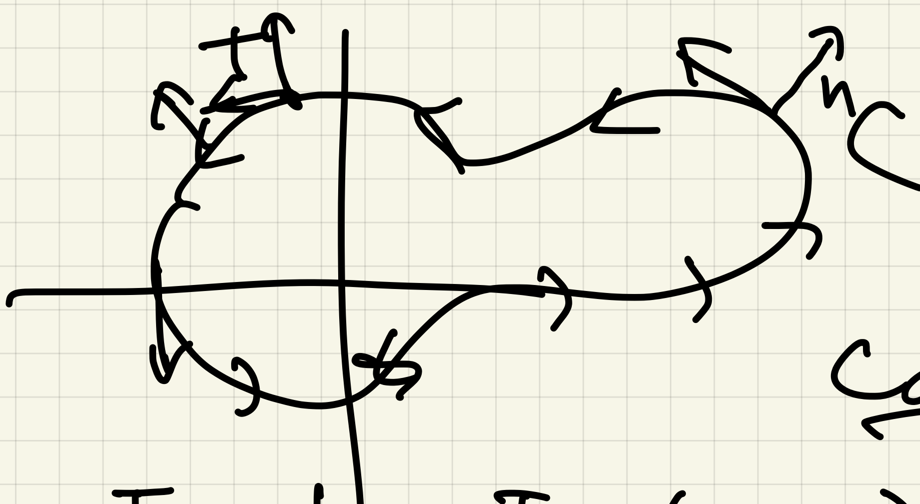


Flux integral of $\vec{F}$ across C =

$$\int_C \vec{F}\cdot\vec{n}\,ds$$

unit normal vector

How to compute it?



$$n = T \times k = T \times (0,0,1)$$

$$\int F\cdot n\,ds = \int F\cdot (T\times k)\,ds$$

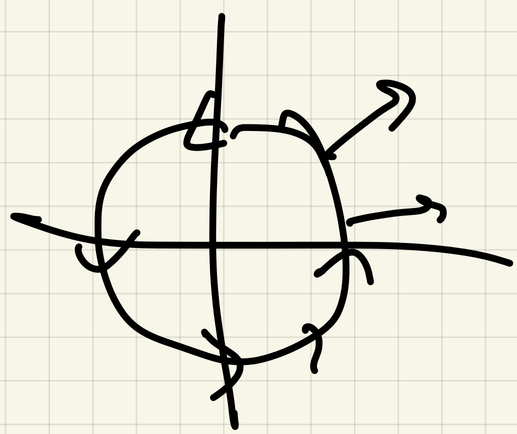
$$\parallel$$

Flux integral $\oint M\,dy - N\,dx$

$\int M\,dx + N\,dy$ work integral

Ex $F = \langle \overset{M}{-y}, \overset{N}{x} \rangle$

$C: r(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle$



should be 0

$$\int_C M\,dy - N\,dx = \int_0^{2\pi} (-\sin t)\underbrace{\cos t}_{\frac{dy}{dt}} - \cos t\underbrace{(-\sin t)}_{\frac{dx}{dt}}\,dt$$

$$= \int_0^{2\pi} 0 \, dt = 0$$