Quiz 13

$$\int_0^3 \underbrace{\int_x^{6-x} x\,dy}\,dx$$

$$xy\Big|_{y=x}^{y=6-x} =$$

$$x(6-x) - x(x) =$$

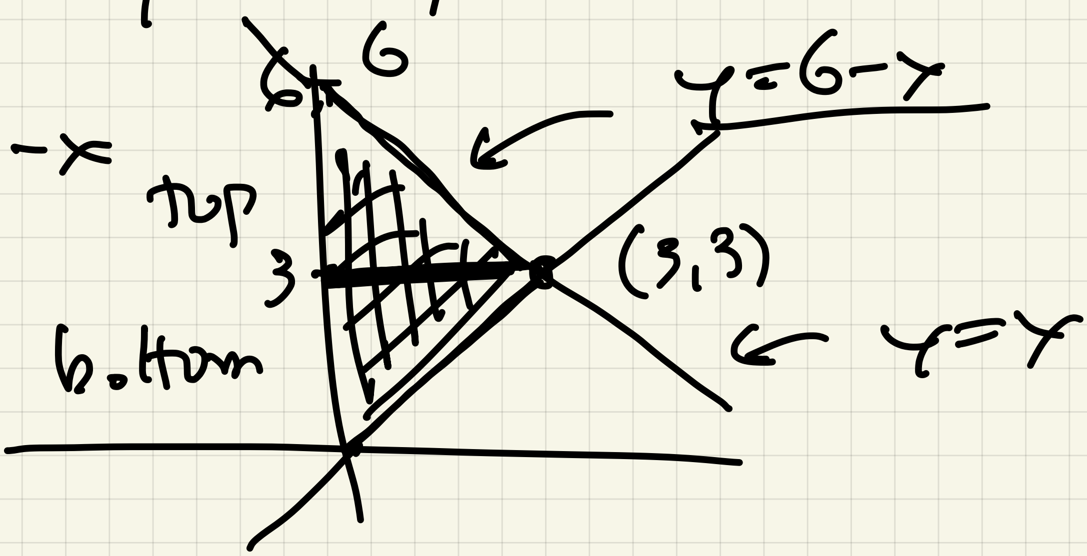
$$\int_0^3 6x - 2x^2\,dx =$$

$$3x^2 - \frac{2}{3}x^3\Big|_0^3 =$$

$$27 - 18 = 9$$

2. $y = 6-x$ / $y = x$

$x \le y \le 6-x$

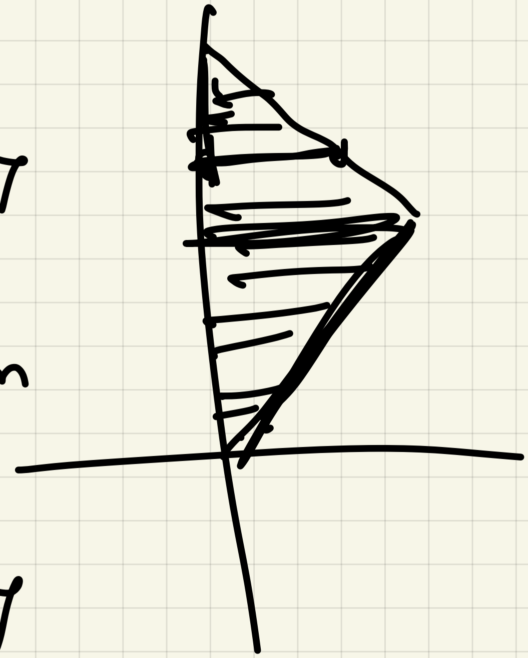


$x = y$

$x = 6 - y$

top

$$\int_3^6 \int_0^{6-y} x \, dx \, dy$$



$+$ bottom

$$\int_0^3 \int_0^y x \, dx \, dy$$

<u>Last time</u> Triple integral

$$\iiint_B f(x,y,z) \, dV$$

$B =$ solid 3D

Compute with iterated integrals

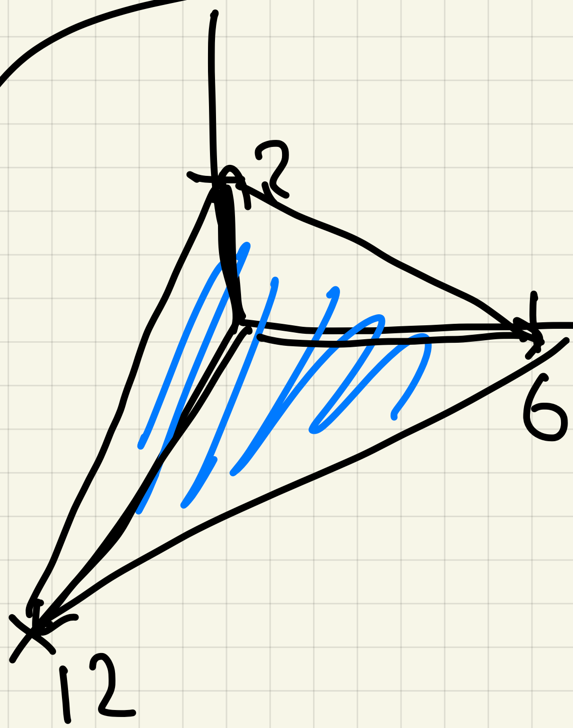
$$\int \int \int f(x,y,z) \, \underline{dz \, dy \, dx}$$

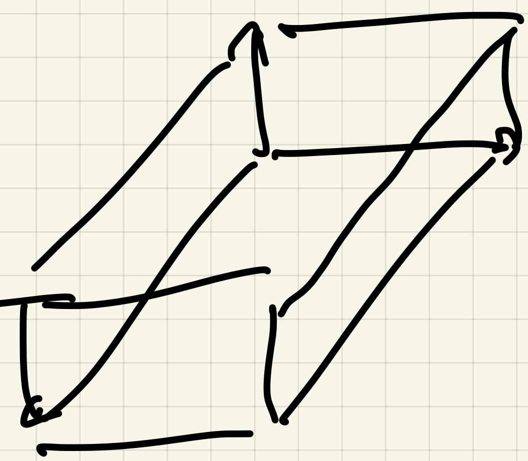
6 orders possible.

Ex1

Find volume of region B in first octant ($x \geq 0, y \geq 0, z \geq 0$)

under plane $\boxed{x+2y+6z=12}$

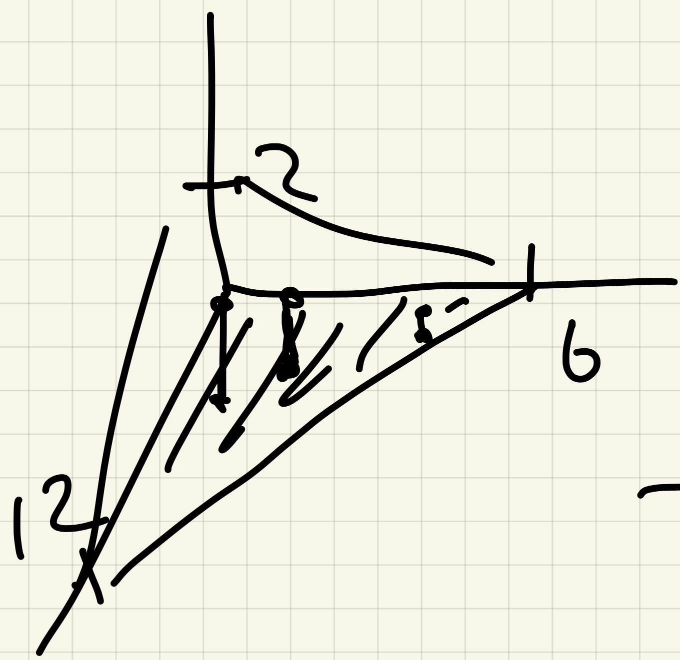


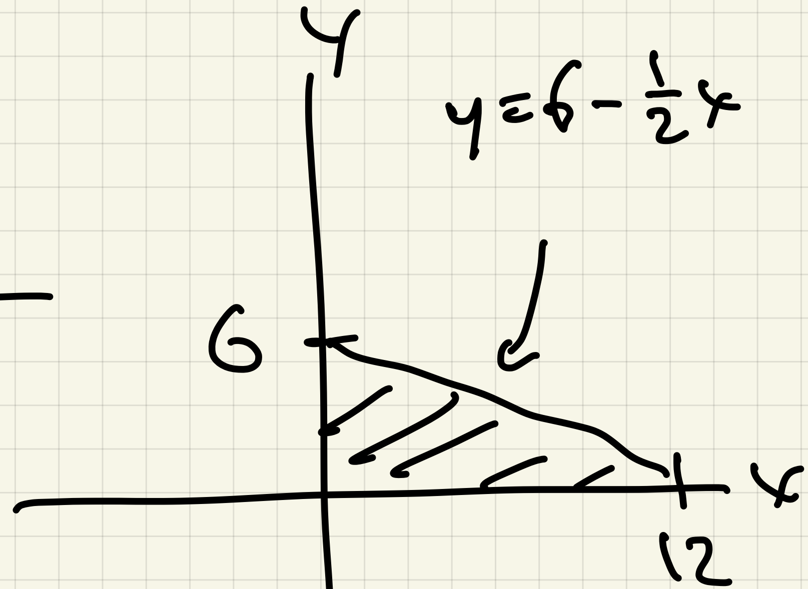


Easy answer: $\frac{1}{6}$ (volume of parallelapiped spanned by $(12,0,0)$, $(0,6,0)$ $(0,0,2)$)

$$\frac{1}{6}(12\cdot 6\cdot 2) = 24$$

$$\left(\iiint 1\,dV \quad [\text{Ex } 5d]\right)$$





$$z = \frac{12-x-2y}{6}$$

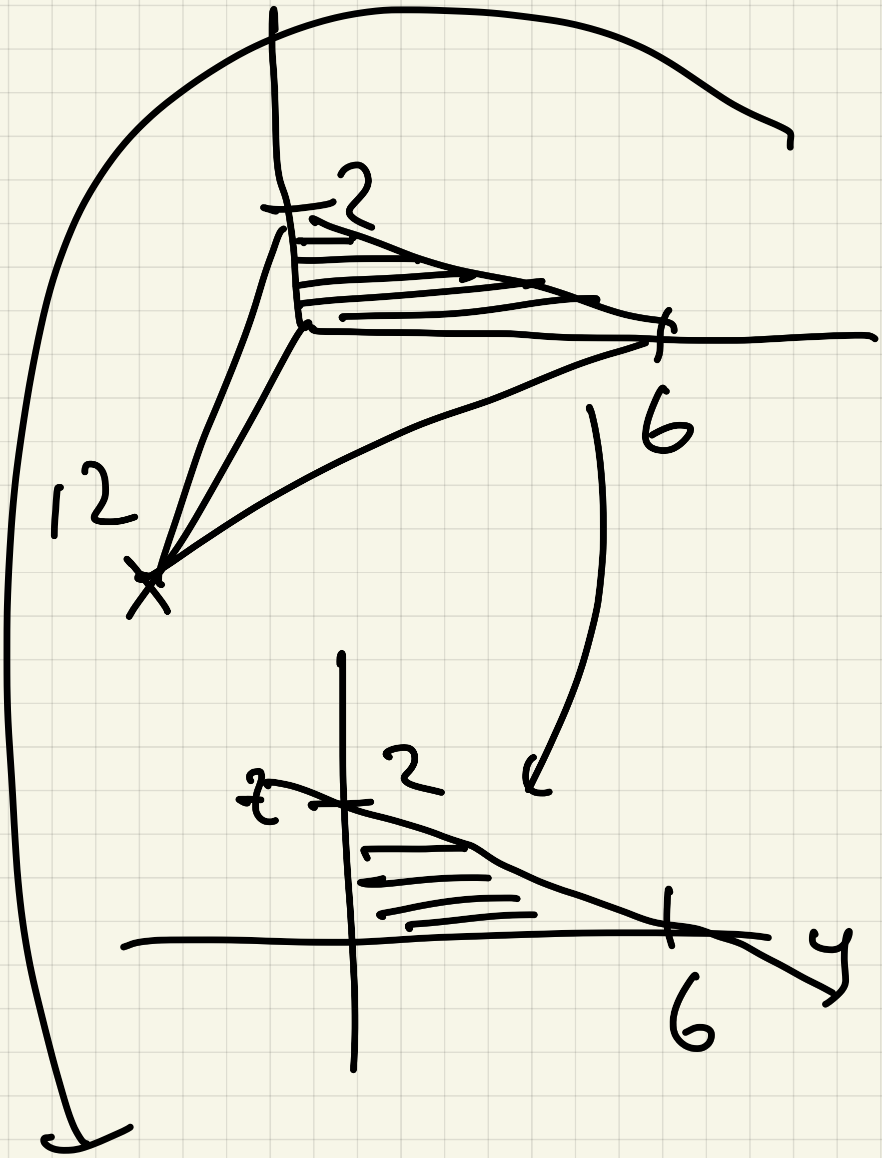
$$0 \le x \le 12$$

$$0 \le y \le 6 - \frac{1}{2}x$$

$$0 \le z \le \frac{12-x-2y}{6}$$

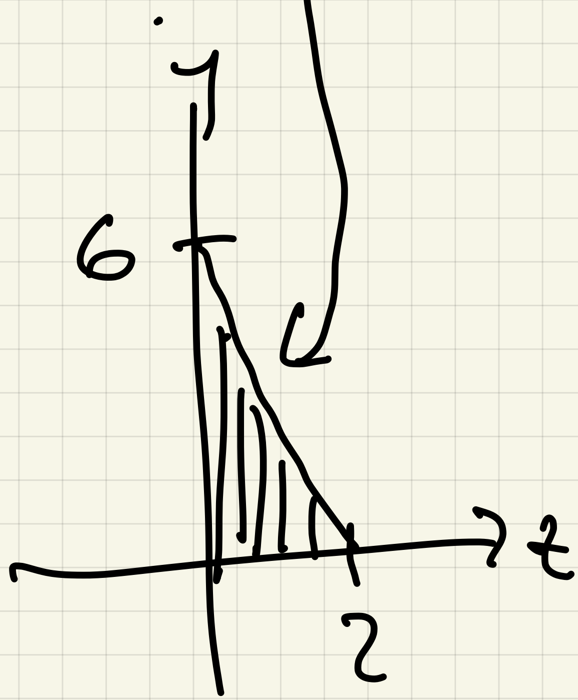
$$\text{Vol} = \int_0^{12}\int_0^{6-\frac{1}{2}x}\int_0^{\frac{12-x-2y}{6}} 1\,dz\,dy\,dx$$

Instead: $dx\ \underline{dy\ dz}$



$0 \le z \le 2$

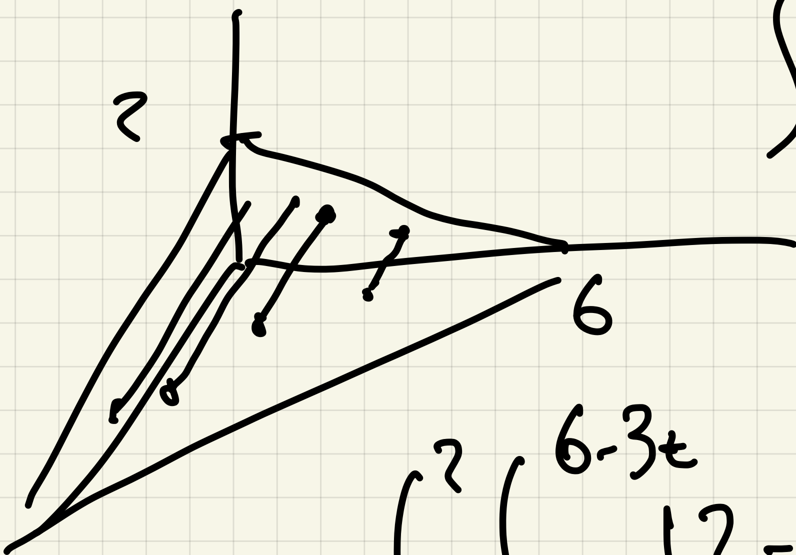
$y = 6 - 3z$



$0 \le z \le 2$

$0 \le y \le 6 - 3z$

$0 \le x \le 12 - 2y - 6z$



$$\int_0^2 \int_0^{6-3z} \underbrace{\int_0^{12-2y-6z} dx}\, dy\, dz$$

$$\parallel$$

$$\int_0^2 \underbrace{\int_0^{6-3z} 12-2y-6z \; dy}\, dz$$

$$12y - y^2 - 6zy \Big|_0^{6-3z} =$$

$$\int_0^2 12(6-3z) - (6-3z)^2 - 6z(6-3z)\, dz$$

$$\int_0^2 (6-3z)\left[\underbrace{12-(6-3z)-6z}_{6-3z}\right] dz$$

$$= \int_0^2 (6-3z)^2\, dz$$

$$\parallel$$

$u = 6-3z$

$du = -3dz$

//

$-\frac{1}{3}du = dz$

$$\int_{u=6}^{u=0} -\frac{1}{3}u^2\,du = \int_0^6 \frac{1}{3}u^2\,du =$$

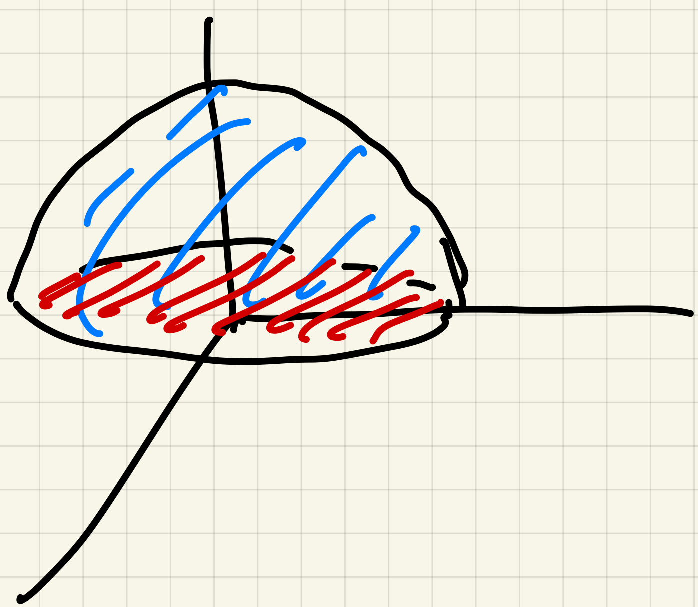
$$\frac{1}{9}u^3 \Big|_0^6 = \frac{216}{9} = 24.$$

<u>Ex 2</u> Find volume of region

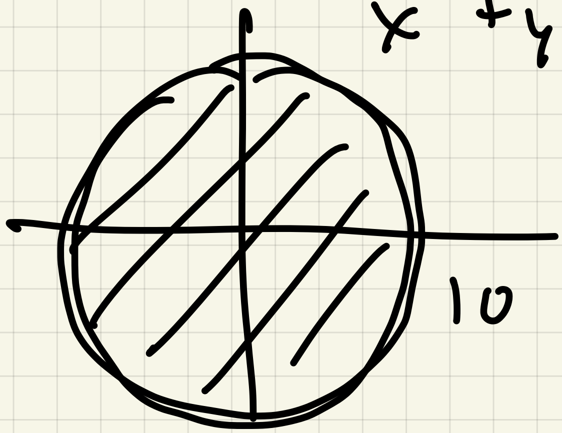
inside sphere

$x^2+y^2+z^2 = 100$

above xy-plane



$x^2+y^2=100 \rightarrow y=\pm\sqrt{100-x^2}$



$-10\le x\le 10$

$-\sqrt{100-x^2}\le y\le\sqrt{100-x^2}$

$0\le z\le\sqrt{100-x^2-y^2}$

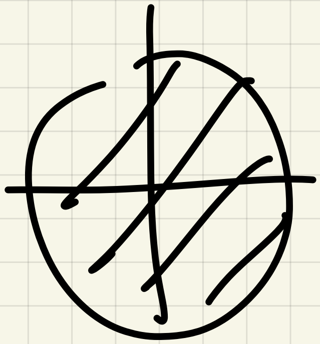
$z=\sqrt{100-x^2-y^2}$

$$V=\int_{-10}^{10}\int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}}\int_{0}^{\sqrt{100-x^2-y^2}} dz\,dy\,dx$$

$$z\Big|_{0}^{\sqrt{100-x^2-y^2}}$$

$$\int_{-10}^{10}\int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}}\sqrt{100-x^2-y^2}\,dy\,dx$$

POLARS!



$$\int_0^{2\pi}\int_0^{10}\sqrt{100-r^2}\,r\,dr\,d\theta$$

$$\|$$

$u = 100 - r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$$-\frac{1}{2}\int_{u=100}^{u=0}\sqrt{u}\,du =$$

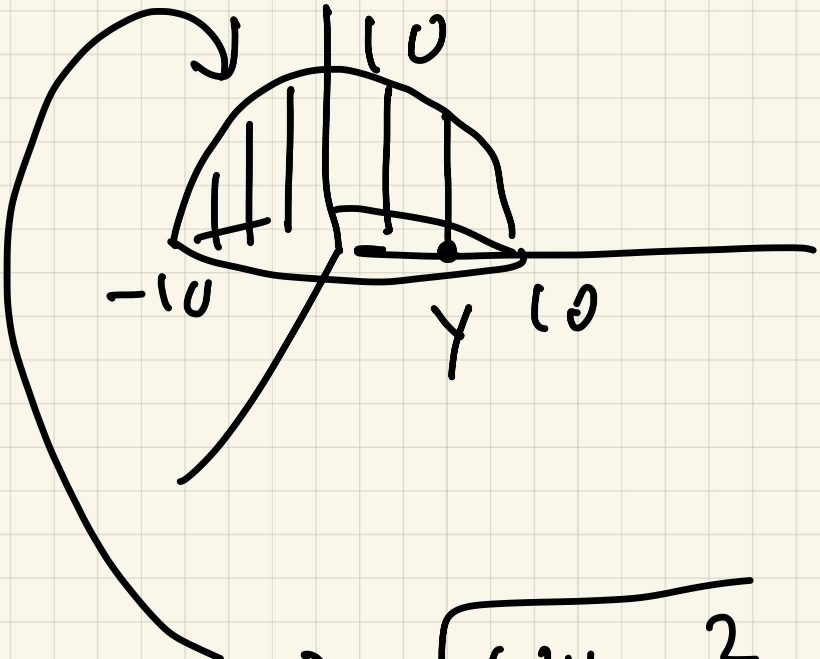
$$+\frac{1}{2}\int_0^{100}\sqrt{u}\,du = \cancel{\frac{1}{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_0^{100} =$$

$$\int_0^{2\pi}\frac{1}{3}\,1000\,d\theta = \frac{1}{3}\,1000\cdot 2\pi =$$

$$\frac{2000\pi}{3}$$

Different order of integration:

$dx\,\underline{dz}\,\underline{dy}$



$-10 \le y \le 10$

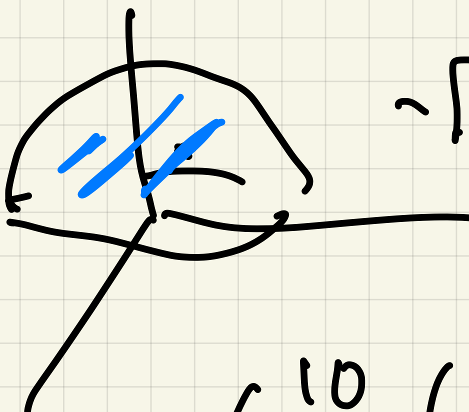
$0 \le z \le \sqrt{100-y^2}$

x

$x^2+y^2+z^2=100$

$z=\sqrt{100-y^2}$

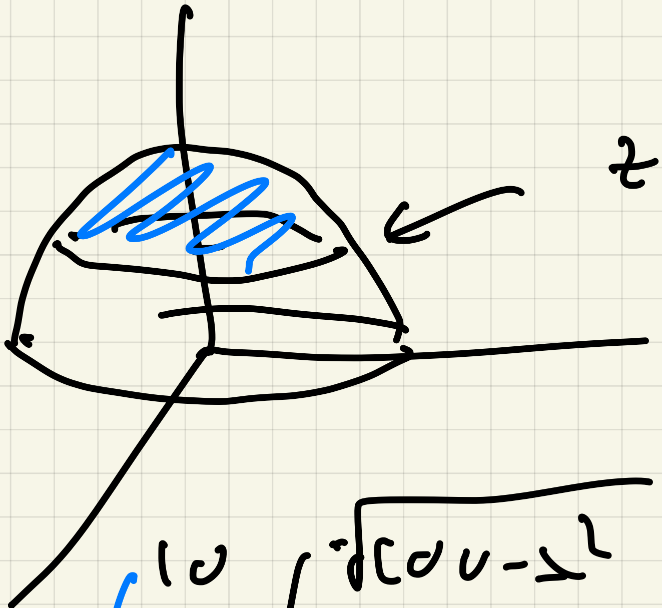
$x=\pm\sqrt{100-y^2-z^2}$



$-\sqrt{100-y^2-z^2} \le x \le \sqrt{100-y^2-z^2}$

so $V = \displaystyle\int_{-10}^{10}\int_{0}^{\sqrt{100-y^2}}\int_{-\sqrt{100-y^2-z^2}}^{\sqrt{100-y^2-z^2}} dx\,dz\,dy$

(c) What about the region inside same sphere, but with $z \geq 5$?



$$\int_{-10}^{10}\int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}}\int_{5}^{\sqrt{100-x^2-y^2}} dz\,dy\,dx$$

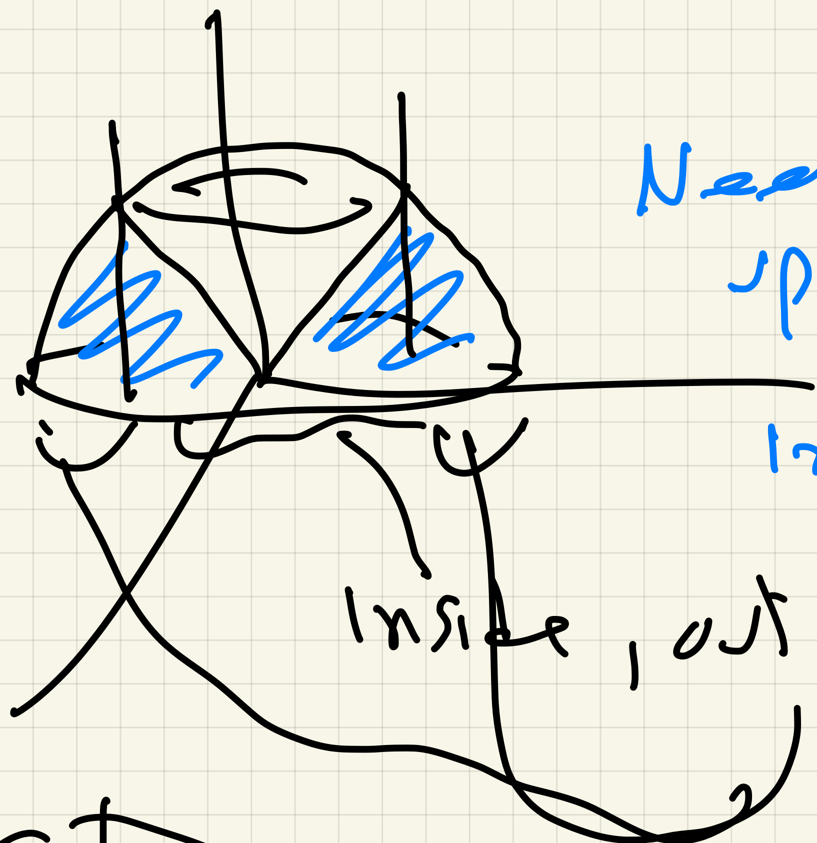
too much

$x^2+y^2+z^2=100$, $z=5$

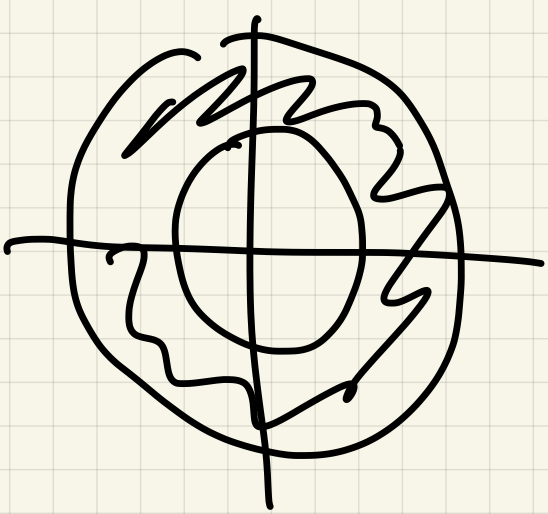
$\boxed{x^2+y^2=75}$ $\quad \sqrt{75}=5\sqrt{3}$

$$\int_{-5\sqrt{3}}^{5\sqrt{3}}\int_{-\sqrt{75-x}}^{\sqrt{75-x^2}}\int_{5}^{\sqrt{100-x^2-y^2}} dz\,dy\,dx$$

(d) What about region inside sphere, above xy plane, outside cone $z = \sqrt{x^2+y^2}$

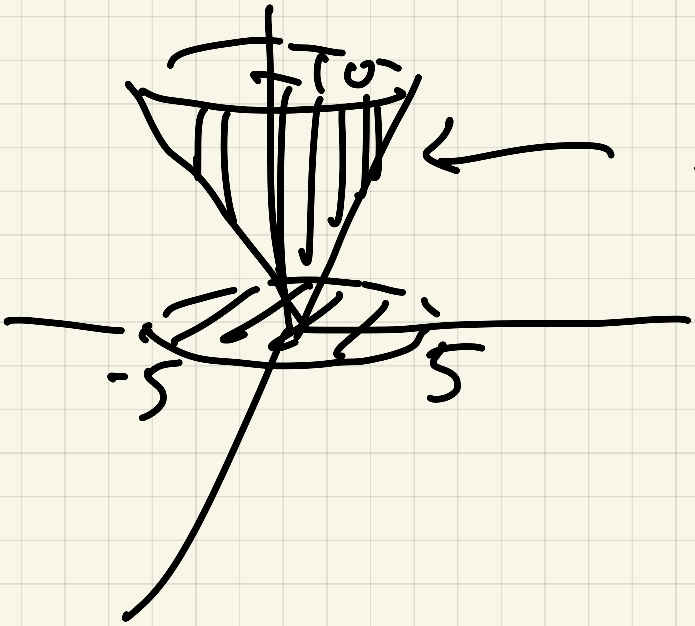


Need to set up two integrals,



xy-range for outside

Ex 3



$$z = 2\sqrt{x^2+y^2}$$

$$V = \frac{1}{3}\pi r^2 h$$

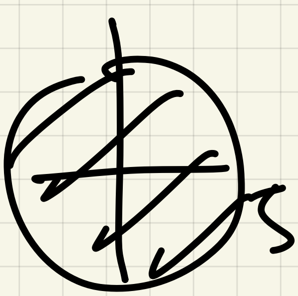
$$= \frac{1}{3}\pi \cdot 5^2 \cdot 10 = \frac{250\pi}{3}$$

$$z = 2\sqrt{x^2+y^2} = 10$$

$$\sqrt{x^2+y^2} = 5 \Rightarrow x^2+y^2 = 25$$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{2\sqrt{x^2+y^2}}^{10} dz\,dy\,dx$$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} 10 - 2\sqrt{x^2+y^2}\,dy\,dx$$



polars:

$$\int_0^{2\pi}\int_0^5 (10-2r)\, r\, dr\, d\theta$$

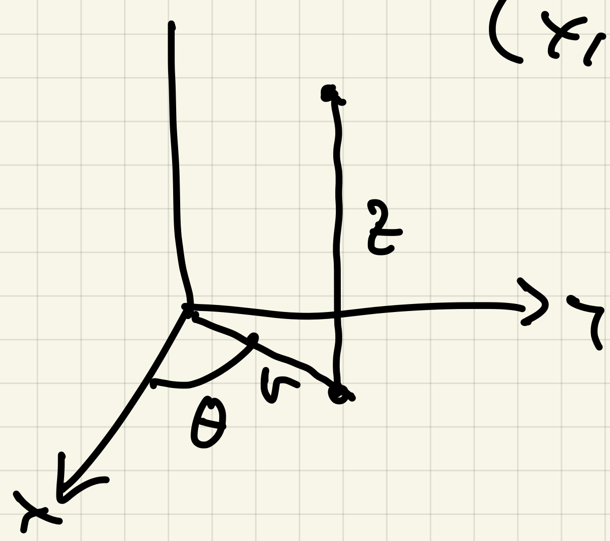
$$10r - 2r^2$$

$$5r^2 - \frac{2}{3}r^3 \Big|_0^5$$

$$125 - \frac{2}{3}(125) = \frac{125}{3}$$

$$\int_0^{2\pi} \frac{125}{3}\, d\theta = \frac{125 \cdot 2\pi}{3} = \frac{250\pi}{3}.$$

Cylindrical coordinates



(x, y, z)

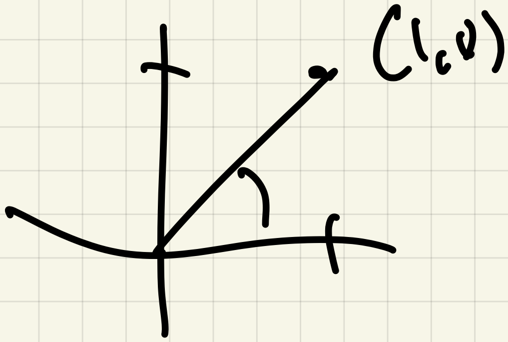
$\updownarrow$

(r, θ, z)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2+y^2=r^2$$

$$\tan\theta = \frac{y}{x}$$

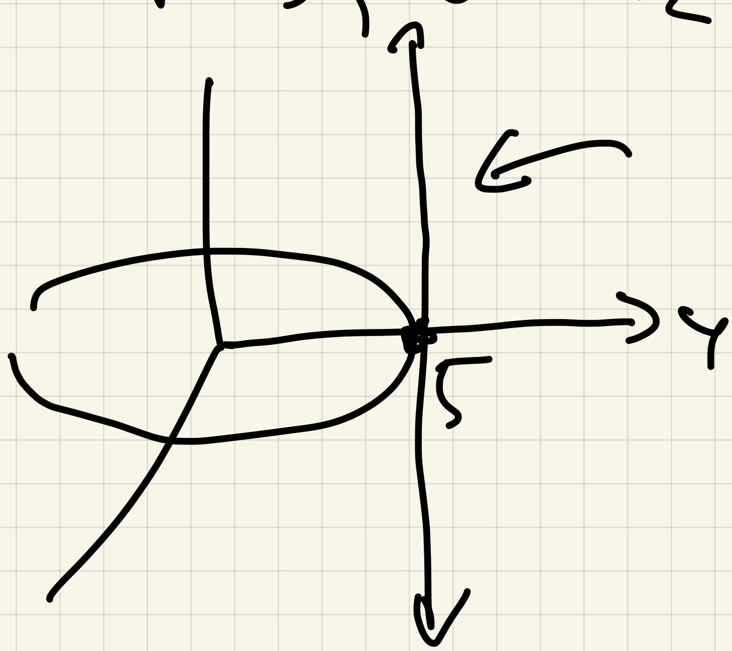
<u>Ex1</u> $(\underset{x}{1},\underset{y}{1},\underset{z}{1}) \sim (\underset{r}{\sqrt{2}}, \underset{\theta}{\tfrac{\pi}{4}}, \underset{z}{1})$



<u>Cylindrical sketches</u>:

<u>Ex2</u> $r = 5, \ \theta = \pi/2$

(a)



line $x = 0$
$y = 5$

(b) $\theta = \pi/4$