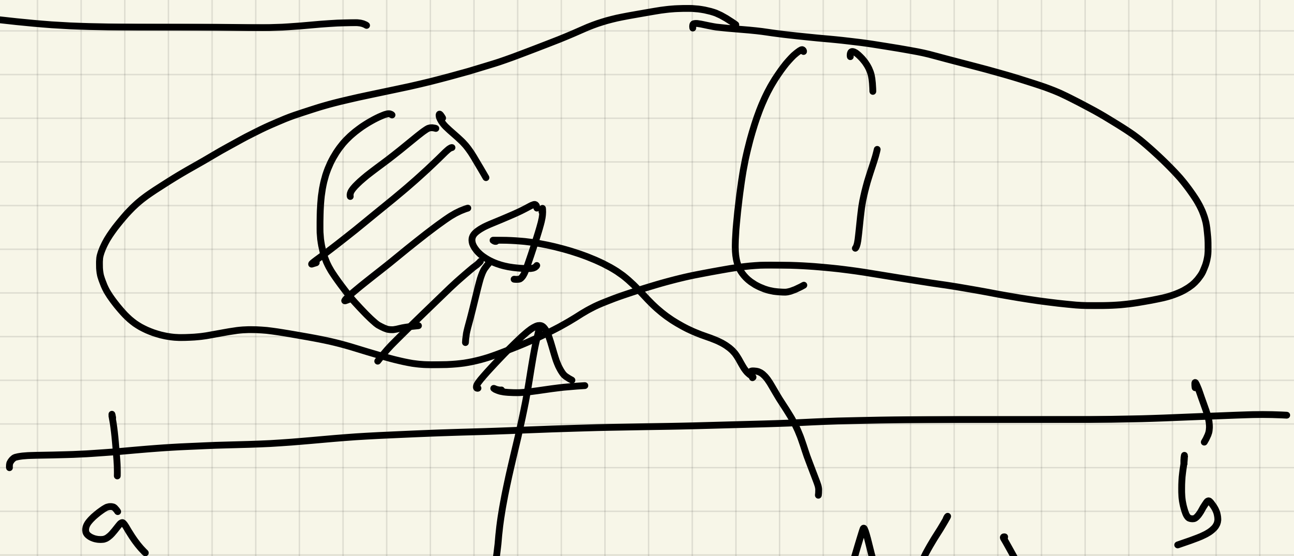
9/3/ Calc3

Main principle

Last time

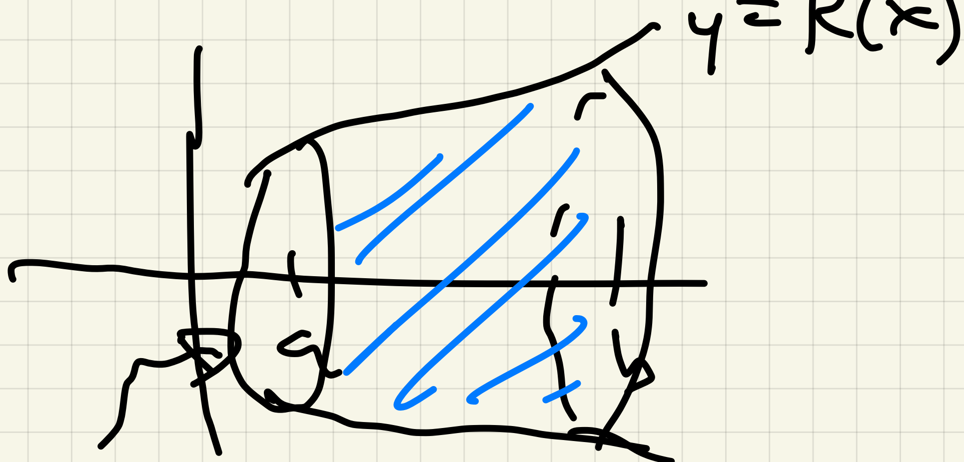



$A(x)$: cross section area

$$\text{Volume} = \int_a^b A(x)\,dx$$

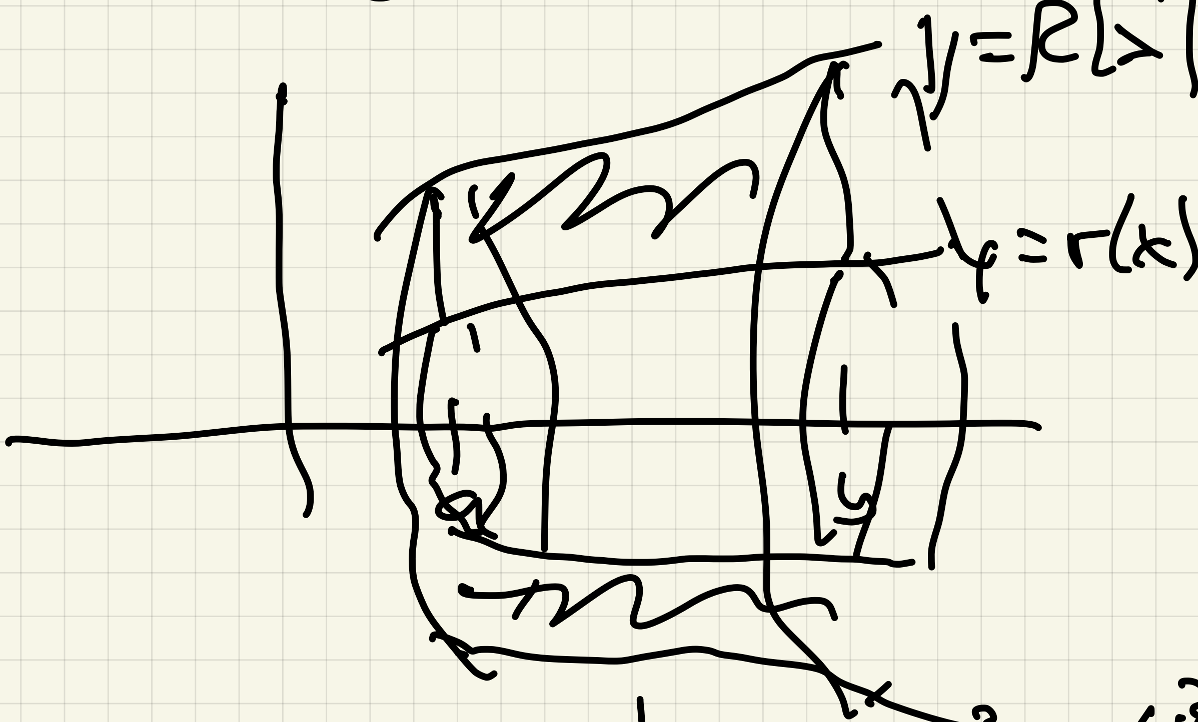
Special cases:

(a)



$$\text{Volume } \int_a^b \pi\,(R(x))^2\,dx$$

(b) "disk method"




$$\text{Volume} = \int_a^b \pi\left(R(x)^2 - r(x)^2\right)dx$$

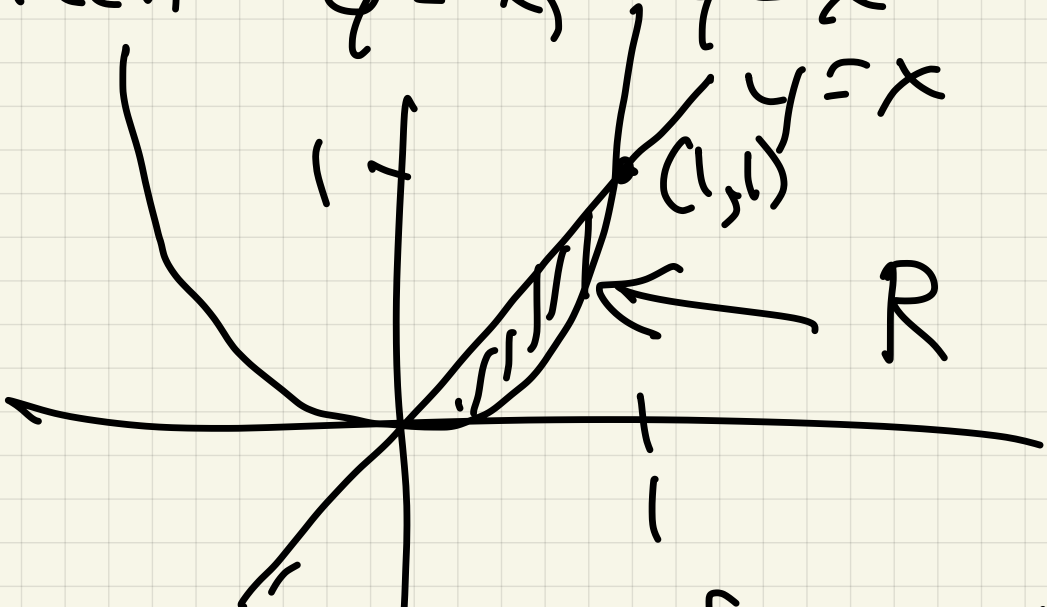
"Washer method"

Ex 1

$$\int_a^b \pi\left(\left(\text{outer rad}\right)^2 - \left(\text{inner rad}\right)^2\right)dx$$

conceptual version

Ex) Let $R$ be the region between $y = x$, $y = x^2$




Find volume of revolution about

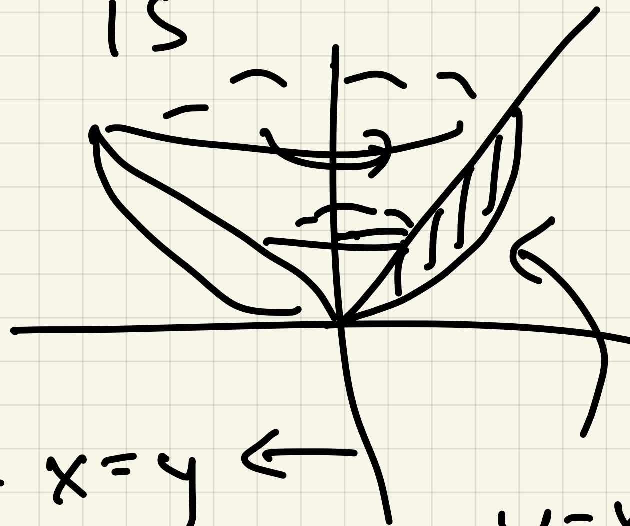
a) $x$-axis
b) $y$-axis
c) line $y=3$
d) line $x=-1$

(a) $V = \int_0^1 \pi\left(x^2 - (x^2)^2\right) dx$

$\pi\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 =$

$$\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{2\pi}{15}$$

(b)



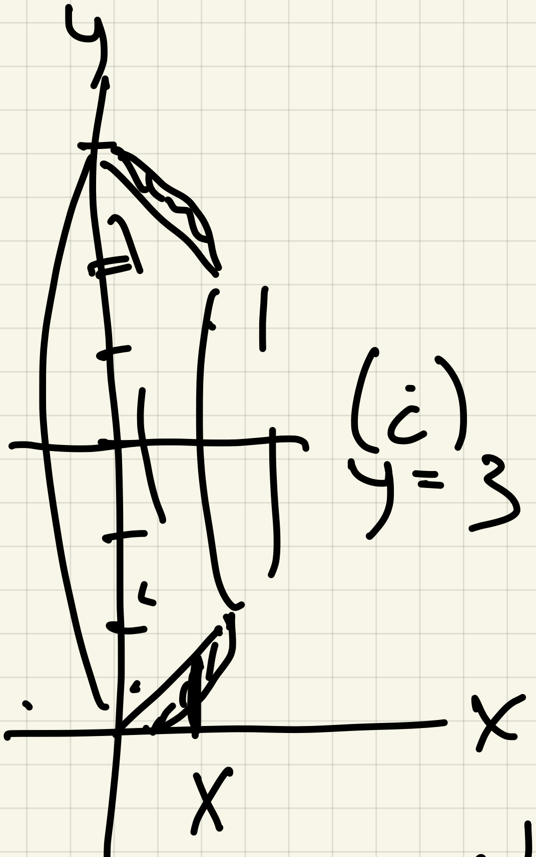
$y=x^2$

inner $= x=y$

outer : $x=\sqrt{y}$

$$V=\int_{y=0}^{y=1}\pi\left(\left(\sqrt{y}\right)^2-y^2\right)dy$$

$$\pi\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_0^1=\frac{\pi}{6}$$




(c) line $y=3$

$y=3$

For $0<x<1$

inner $r=3-x$

outer $R=3-x^2$

$$V=\int_0^1\pi\left(\left(3-x^2\right)^2-\left(3-x\right)^2\right)dx$$

$$\int_0^1 \pi\left(\left(\cancel{9}-\underline{\underline{6x^2}}+\underline{\underline{x^4}}\right)-\left(\cancel{9}-\underline{\underline{6x}}+\underline{\underline{x^2}}\right)\right)dx$$

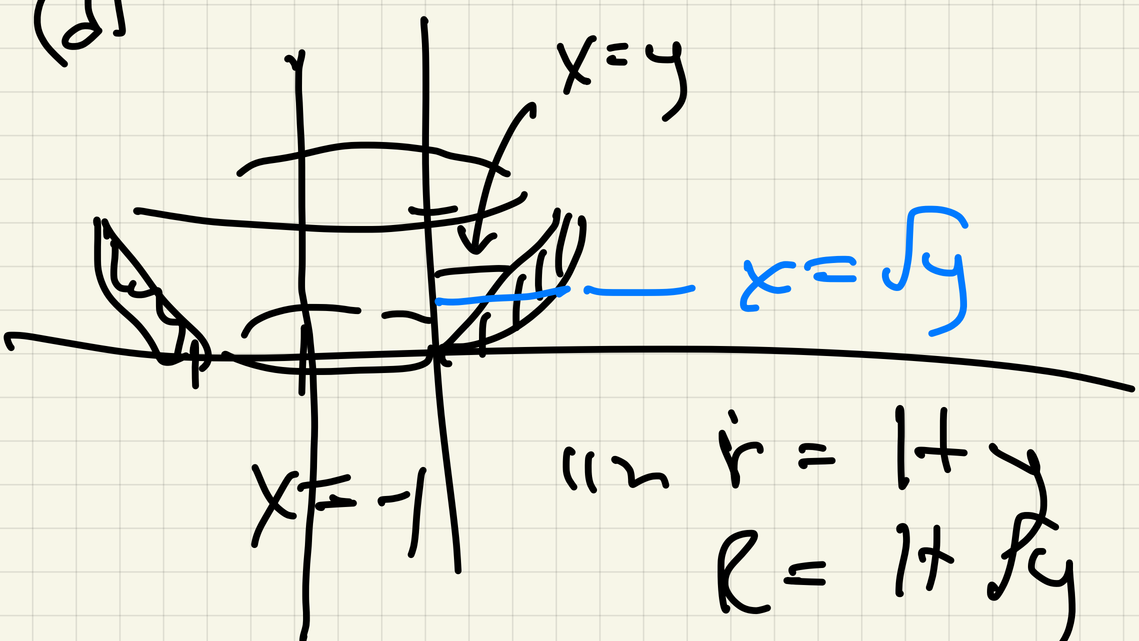
$$\int_0^1 \pi\left(x^4-7x^2+6x\right)dx =$$

$$\pi\left(\frac{x^5}{5}-\frac{7}{3}x^3+3x^2\right)\Big|_0^1 =$$

$$\pi\left(\frac{1}{5}-\frac{7}{3}+3\right) = \pi\left(\frac{3-35+45}{15}\right) =$$

$$13\pi/15$$

(d)




$$r = 1+y$$

$$R = 1+\sqrt{y}$$

$$V = \int_0^1 \pi\left(\left(\underline{\underline{1+\sqrt{y}}}\right)^2-(1+y)^2\right)dy$$

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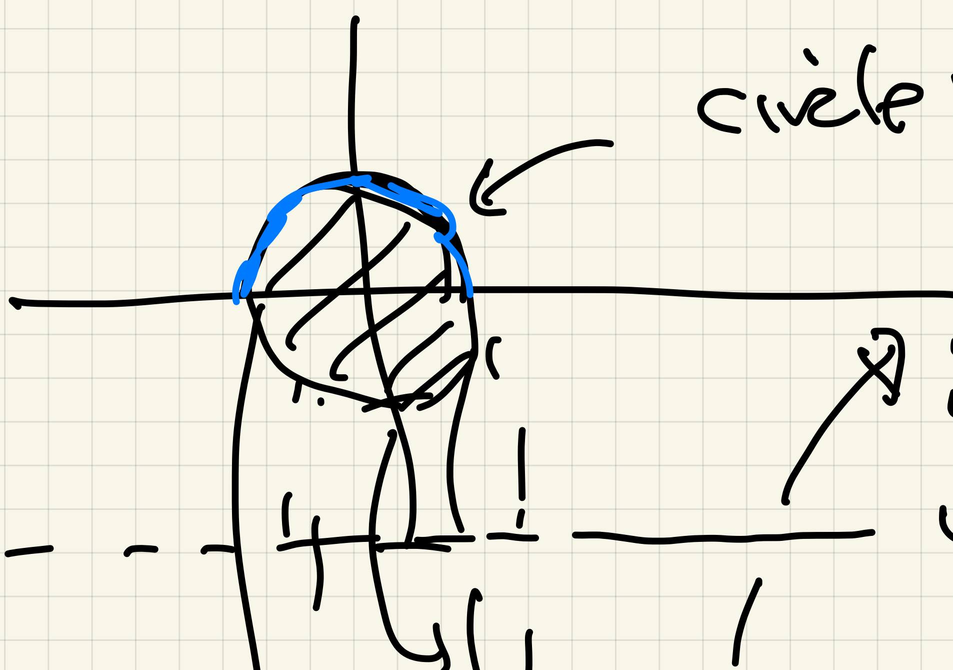
$$\int_0^1 \pi \left(\cancel{1}+\underline{2\sqrt{y}}+y\right)-\left(\cancel{1}+2y+y^2\right)$$

$$\pi\int_0^1 2\sqrt{y}-y-y^2\,dy =$$

$$\pi\ \frac{4}{3}y^{3/2}-\frac{1}{2}y^2-\frac{1}{3}y^3\Big|_0^1 =$$

$$\pi\left(\frac{4}{3}-\frac{1}{2}-\frac{1}{3}\right)=\frac{\pi}{2}$$

<u>Ex 2</u> Find volume of revolution of disk $x^2+y^2\le 1$ about line $y=-2$.

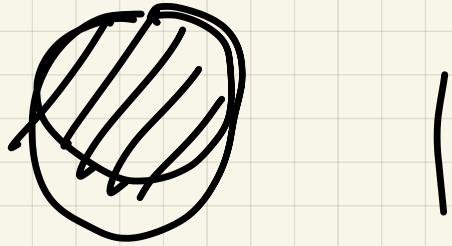


circle: $x^2+y^2=1$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

$y=-2$



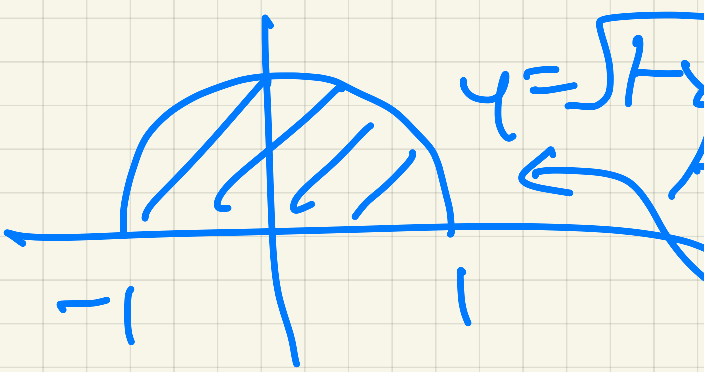
outer $= 2+\sqrt{1-x^2}$

inner $= 2-\sqrt{1-x^2}$

$$V = \int_{-1}^{1} \pi\left(\left(2+\sqrt{1-x^2}\right)^2 - \left(2-\sqrt{1-x^2}\right)^2\right)dx$$

$$= \int_{-1}^{1} \pi\left(\cancel{4}+4\sqrt{1-x^2}+(1-\cancel{x^2})\right) - \left(\cancel{4}-4\sqrt{1-x^2}+(1-\cancel{x^2})\right)dx$$

$$= \pi\int_{-1}^{1} 8\sqrt{1-x^2}\,dx = 8\pi\underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx}$$




$y=\sqrt{1-x^2}$

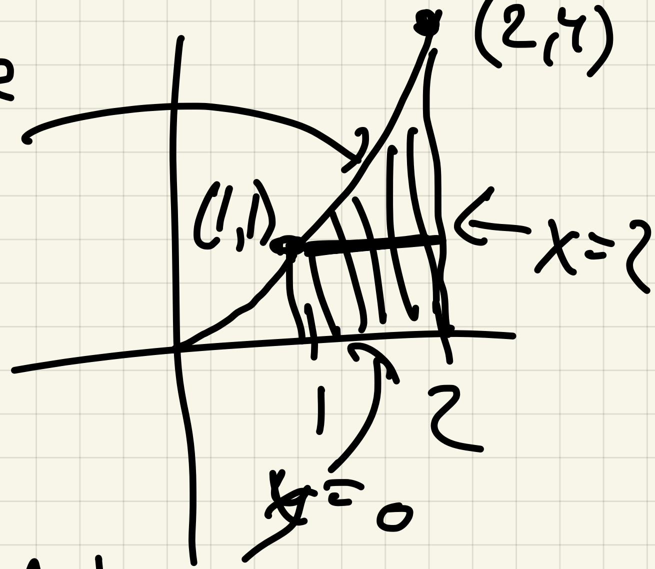
$\leftarrow$ Area $= \pi/2$

$$8\pi\cdot\frac{\pi}{2} = 4\pi^2$$

<u>Ex 3</u> Revolve about y-axis

and find volume

$y=x^2$




$y=x^2$

$$\text{Volume} = \text{Vol}_{top} + \text{Vol}_{bottom}$$

$$\text{Vol}_{top} = \int_{y=1}^{y=4} \pi\left(2^2 - (\sqrt{y})^2\right) dy$$

$$= \pi \int_1^4 4-y \; dy$$

$$= \pi\left(4y - \frac{y^2}{2}\right)\Bigg|_1^4 =$$

$$\pi\left(8 - \frac{7}{2}\right) = \frac{9\pi}{2}$$

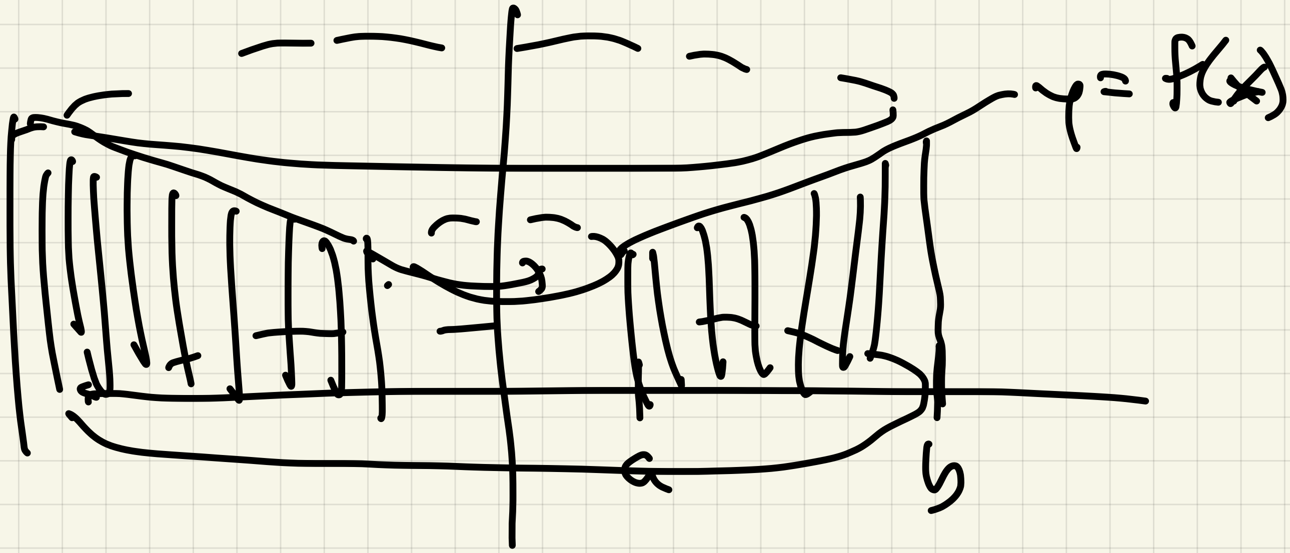
$$\text{Vol}_{bottom} = \int_{y=0}^{y=1} \pi\left(2^2 - 1^2\right) dy = 3\pi$$

X

$$\frac{9\pi}{2} + \frac{6\pi}{2} = \frac{15\pi}{2}.$$

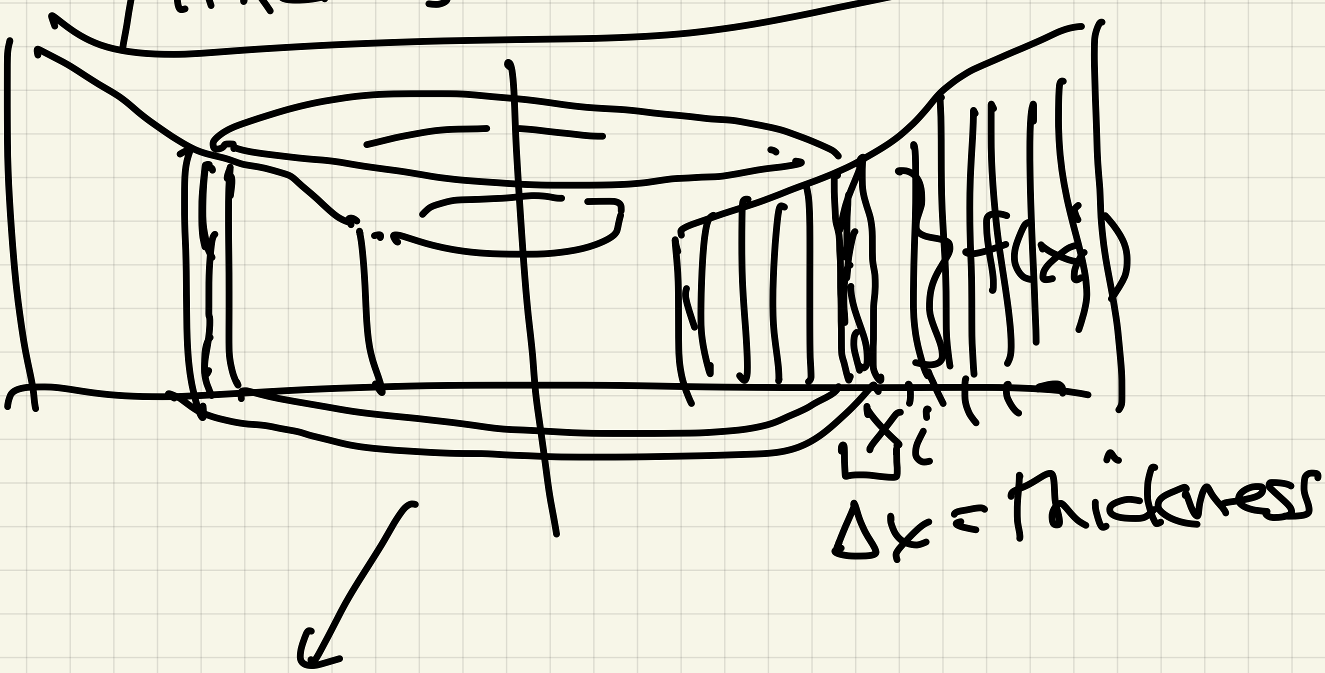
§6.1 → §6.2

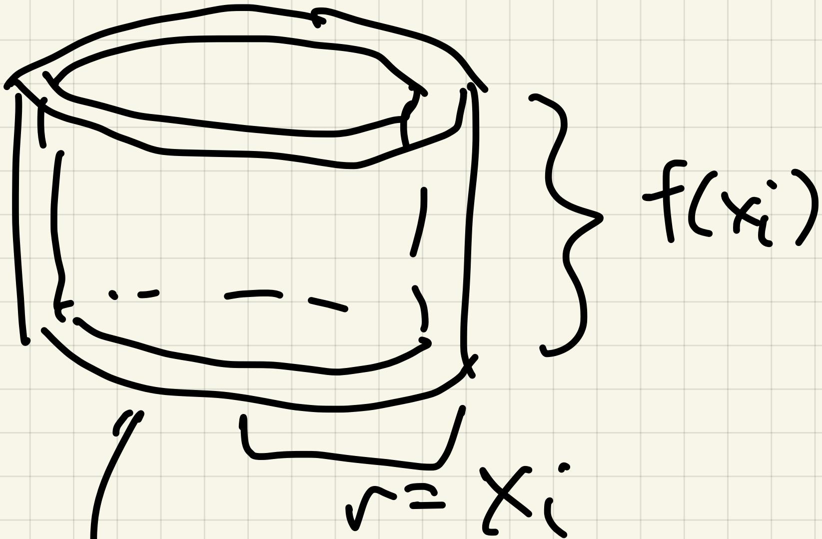
# Another Idea: shell method



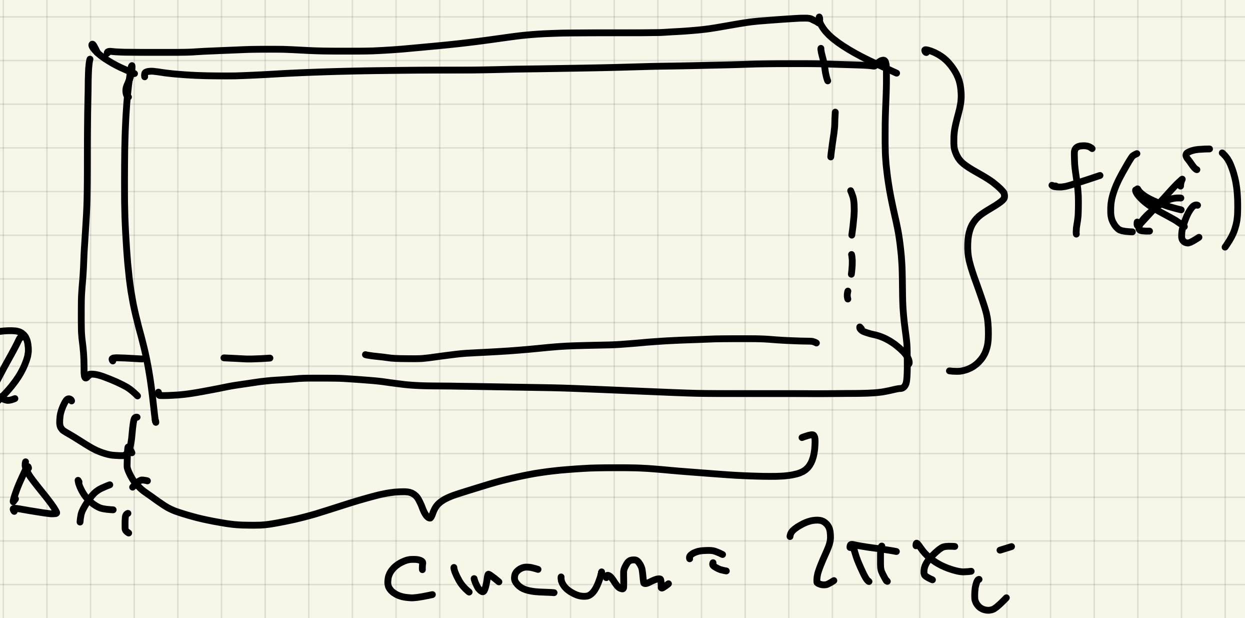

Let $V$ = volume:

Can estimate $V$ by cutting the solid into thick cylinders:






cut along edge, flatten




Volume = $2\pi x_i \cdot f(x_i) \Delta x_i$

so Total volume

$$V \approx \sum_{i=1}^{n} 2\pi x_i \cdot f(x_i) \Delta x_i$$

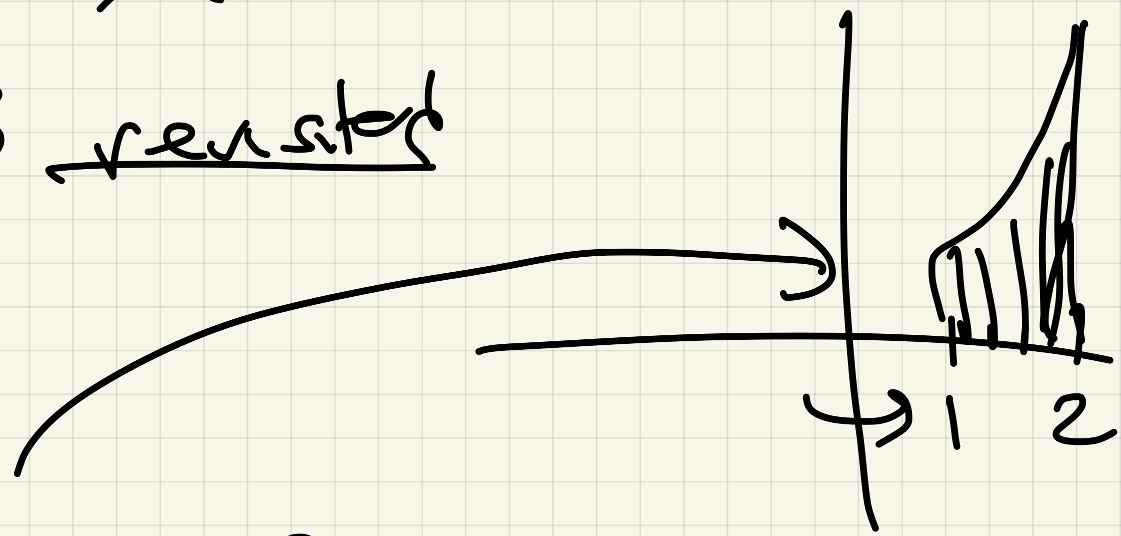
So **Exact Volume** =

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x_i =$$

← Riemann sum

$$\int_a^b 2\pi x f(x)\,dx$$

**Ex 3 revisited**




Volume of revolution is

$$\int_1^2 2\pi x(x^2)\,dx = \int_1^2 2\pi x^3\,dx =$$

$$4 \quad \frac{\pi}{2}x^4\Big|_1^2 =$$

$$\pi/2\,(2^4-1^4) = 15\pi/2$$