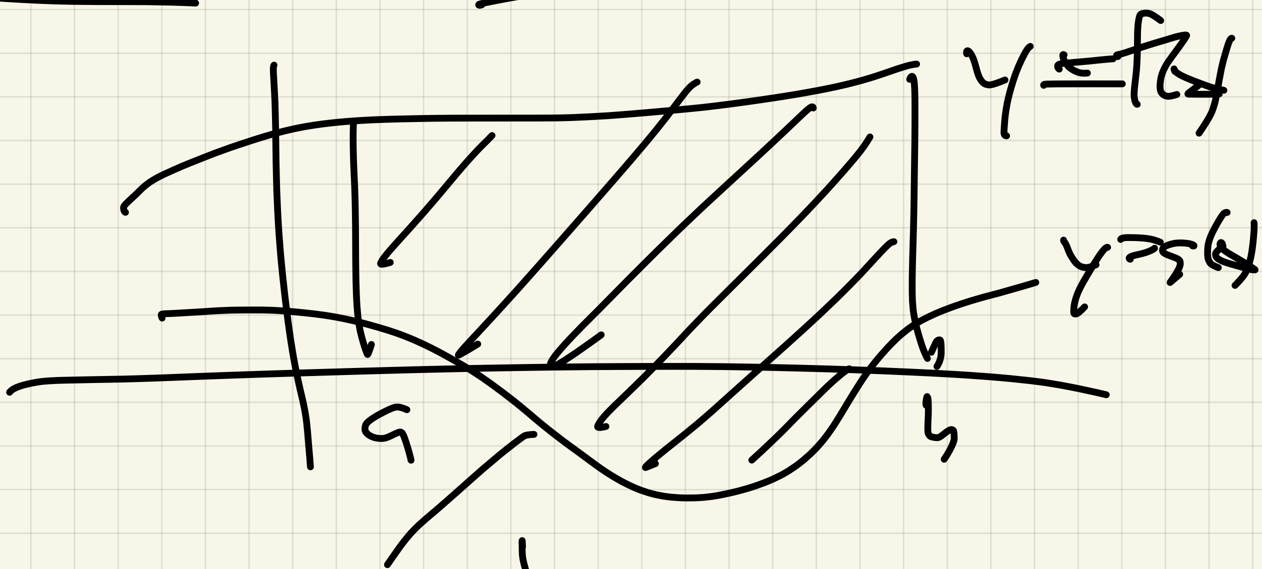
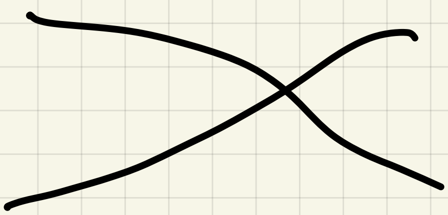
9/11 Calc 2

Last time

Area between curves:

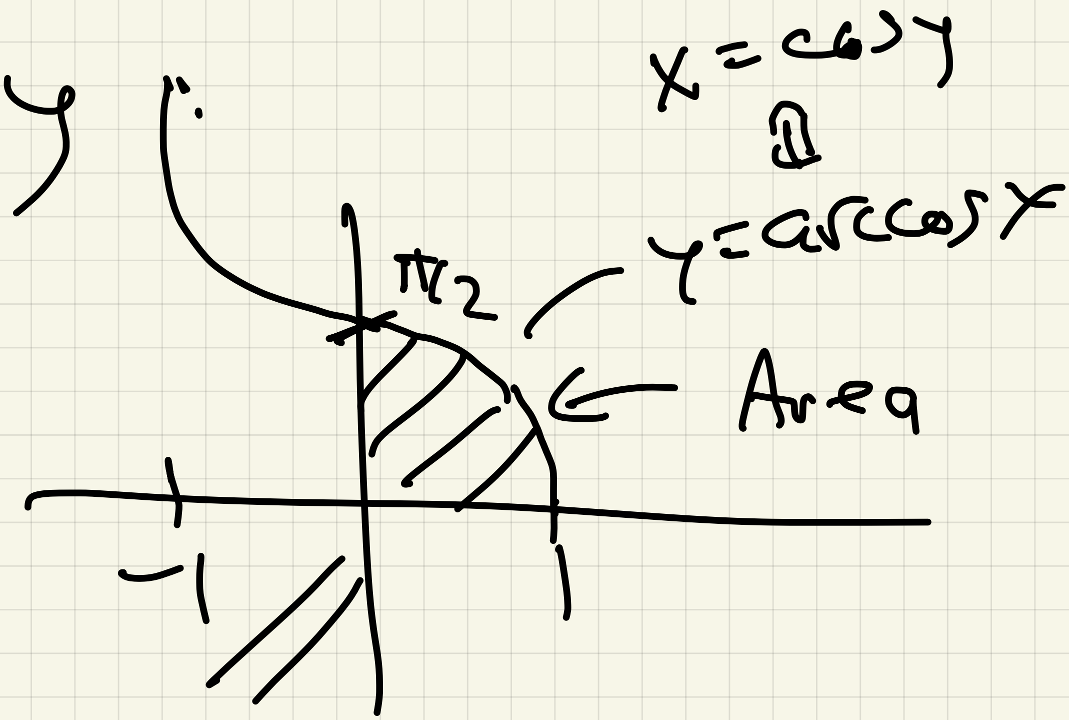


$$\text{Area} = \int_a^b f(x) - g(x)\,dx$$



Also can switch roles x & y:

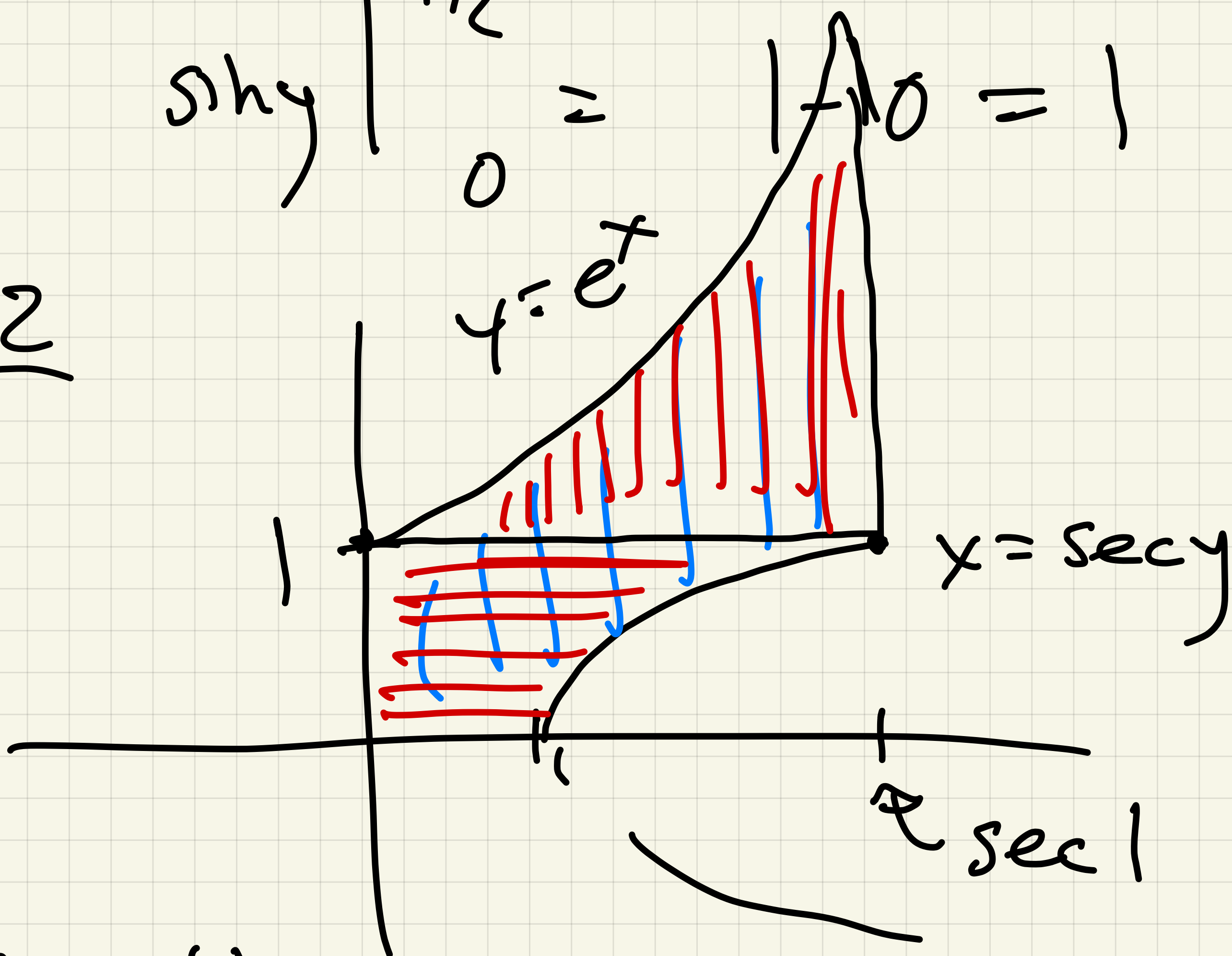
Ex 1



$$\int_0^1 \arccos x \, dx = \int_0^{\pi/2} \cos y \, dy$$

$$= \sin y \Big|_0^{\pi/2} = 1 - 0 = 1$$

Ex 2



Find blue area

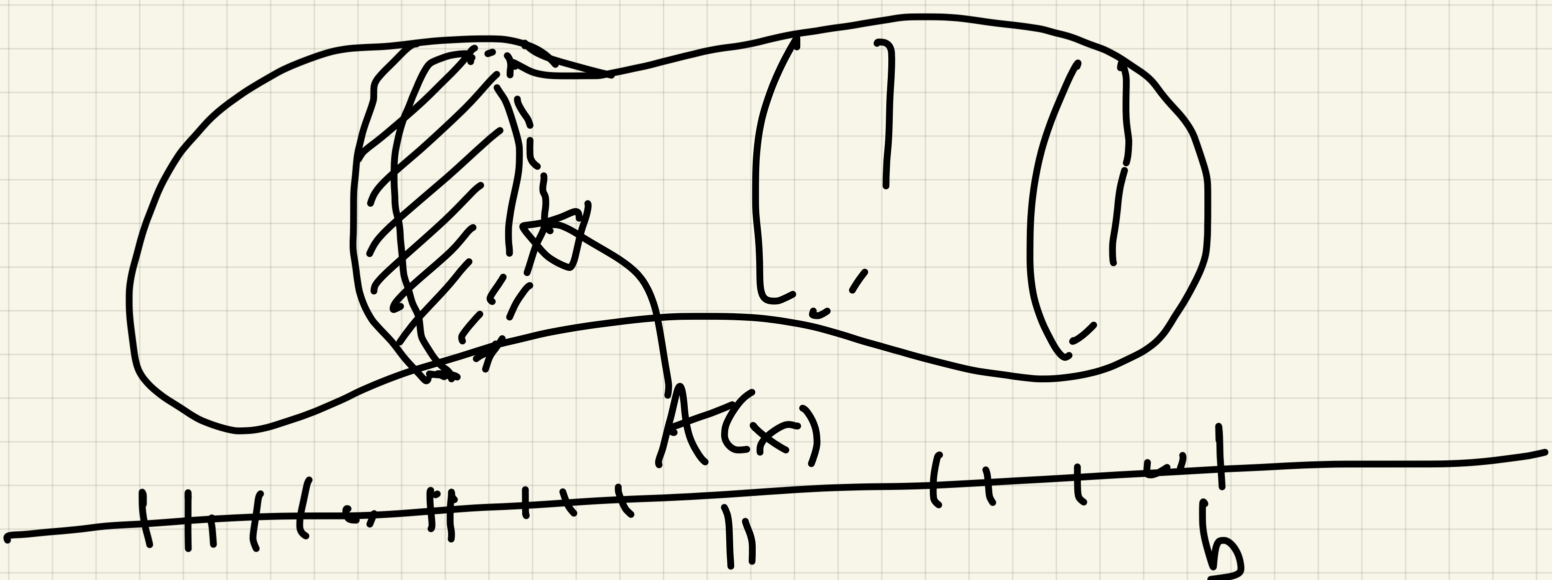
Area = Area top ($y > 1$) + Area bottom ($y < 1$)

$$= \int_0^{\sec 1} e^x - 1 \, dx + \int_0^1 \sec y \, dy$$

$$= e^x - x \Big|_0^{\sec 1} + \ln|\sec y + \tan y| \Big|_0^1$$

$$= e^{\sec 1} - \sec 1 - 1 + \ln|\sec 1 + \tan 1|)$$

<u>Volume</u> :   potato



cross-sectional area

<u>Idea</u>:

If $A(x)$ is cross sectional area at position x, we can estimate volume

$$\text{Volume} \approx \sum_{i=1}^{n} \begin{pmatrix} \text{Volume} \\ \text{of slice } i \end{pmatrix} \approx$$

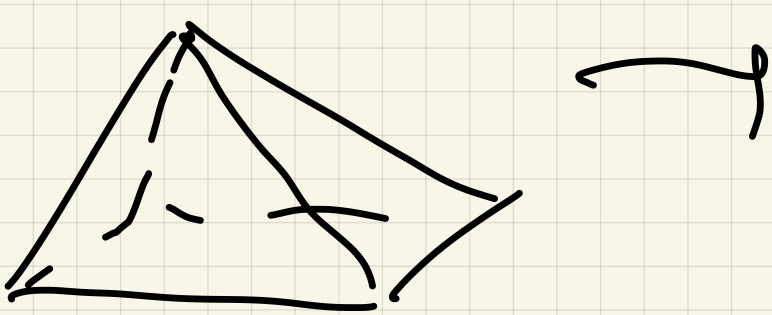
thickness of slice

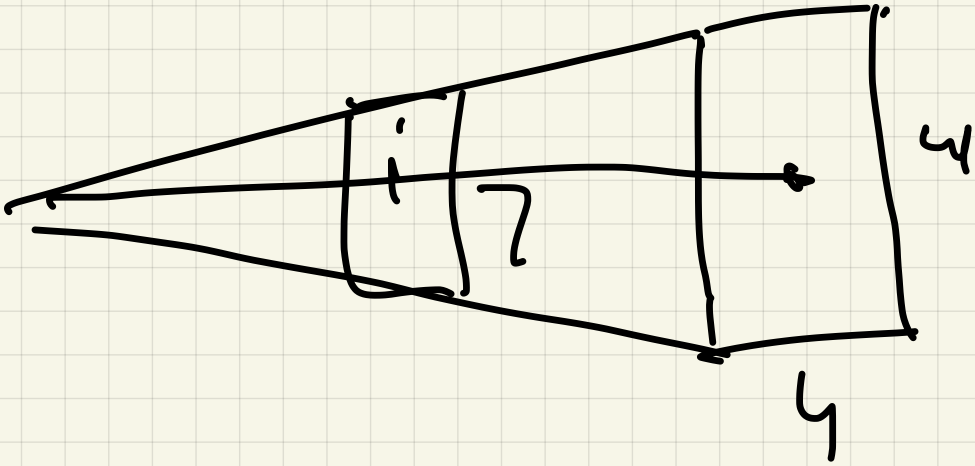
$$\sum_{i=1}^{n} A(x_i) \cdot \Delta x_i$$

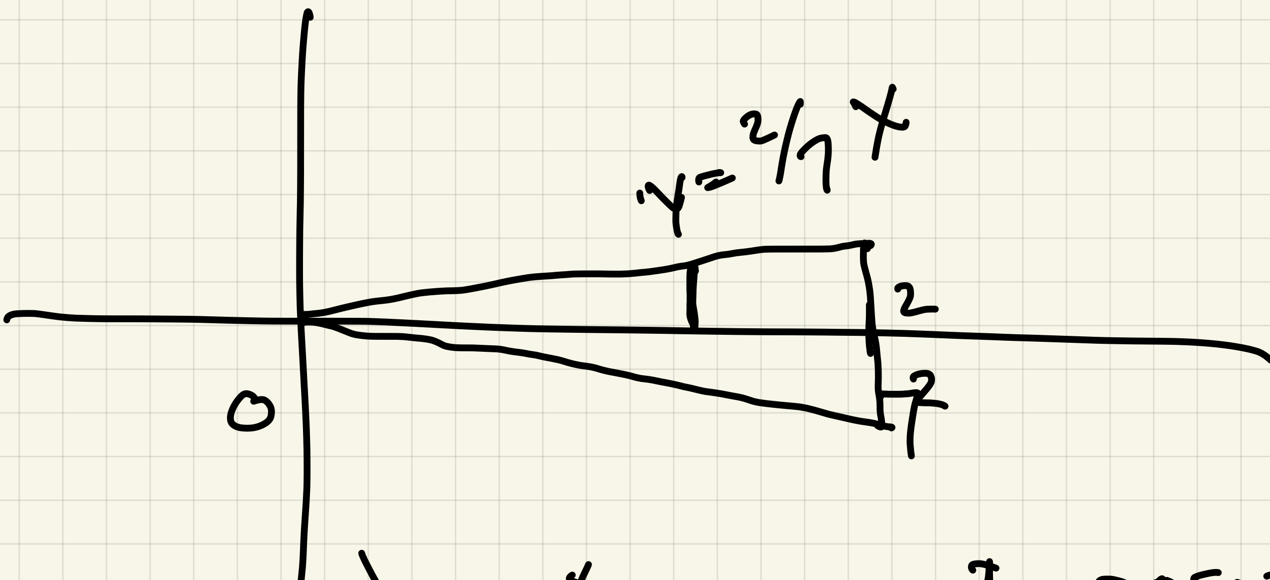
$\therefore$ Exact volume is

$$\lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x_i = \int_a^b A(x)\,dx$$

<u>Ex1</u> Find volume of pyramid of height 7 with square 4×4 base







so side of square at position x

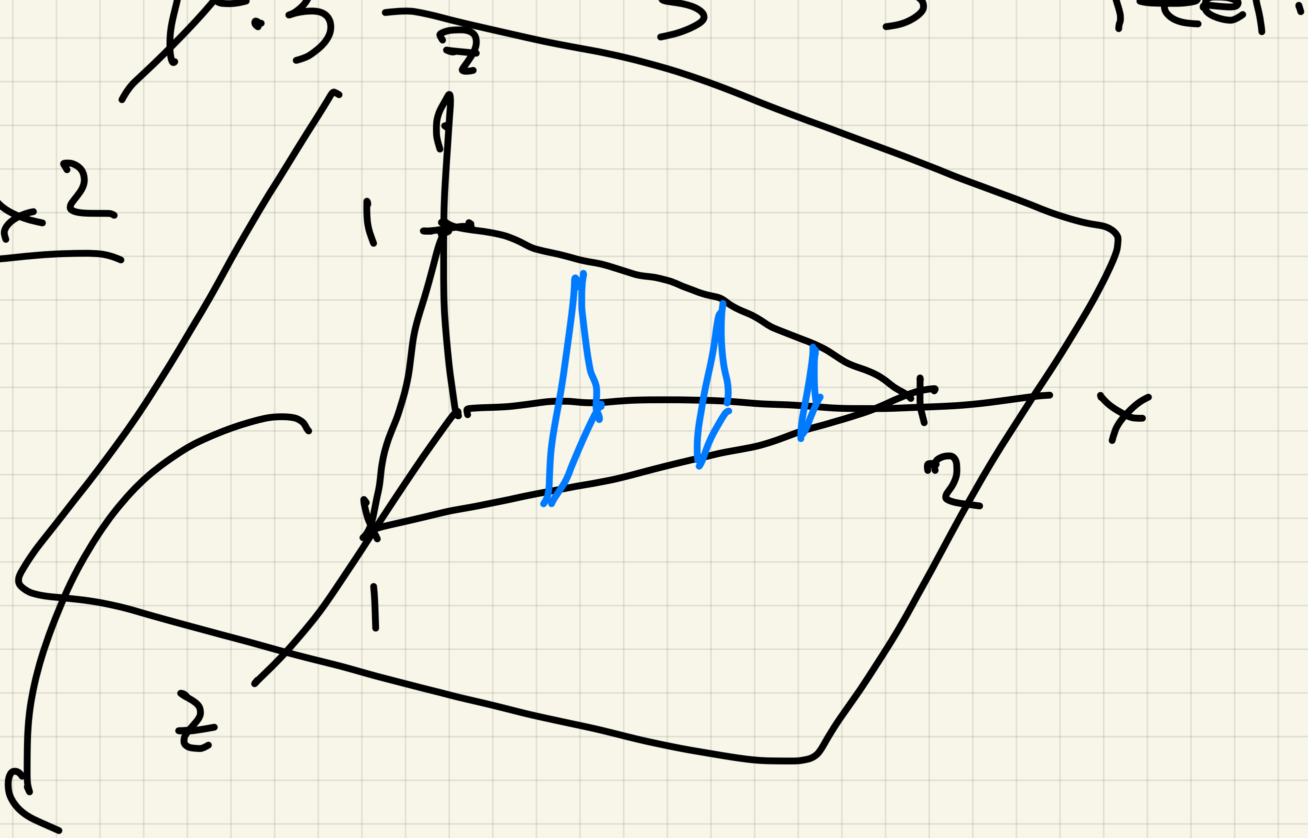
has length $2\left(\frac{2}{7}x\right) = \frac{4x}{7} = s$

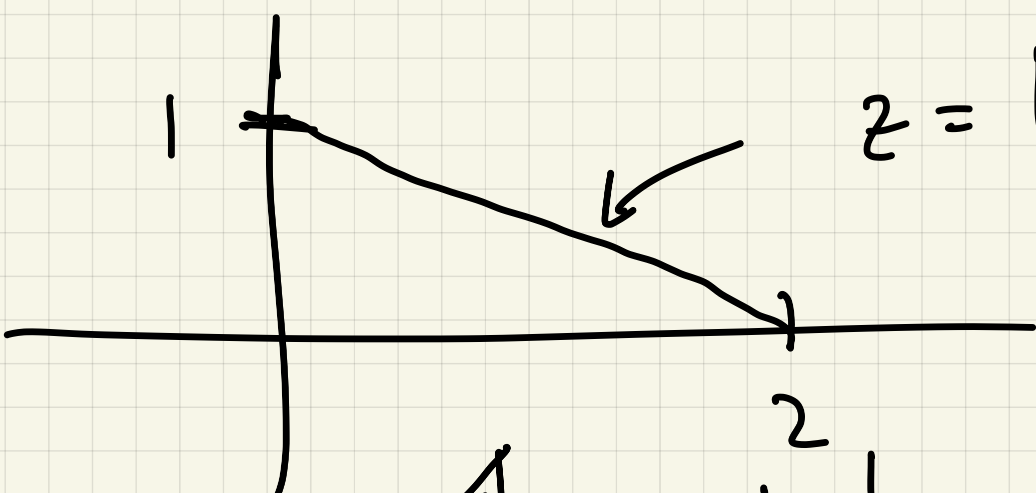
so $A(x) = s^2 = \left(\frac{4x}{7}\right)^2 = \frac{16x^2}{49}$

so $V = \int_0^7 \frac{16x^2}{49} = \frac{16}{49}\frac{x^3}{3}\Big|_0^7 =$

$$\frac{16\cdot 7^{\cancel{3}}}{\cancel{7^2}\cdot 3} = \frac{16\cdot 7}{3} = \frac{112}{3} \text{ cubic feet.}$$

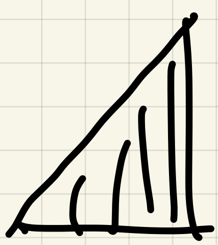
Ex 2





$z = 1 - \frac{1}{2}x$

so



$1 - \frac{1}{2}x$

$1 - \frac{1}{2}x$

so $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(1 - \frac{1}{2}x\right)^2$

So $V = \int_{x=0}^{x=2} A(x)\,dx =$

$$\int_0^2 \frac{1}{2}\left(1-\frac{1}{2}x\right)^2 dx$$

$$\boxed{u = 1-\frac{1}{2}x}$$

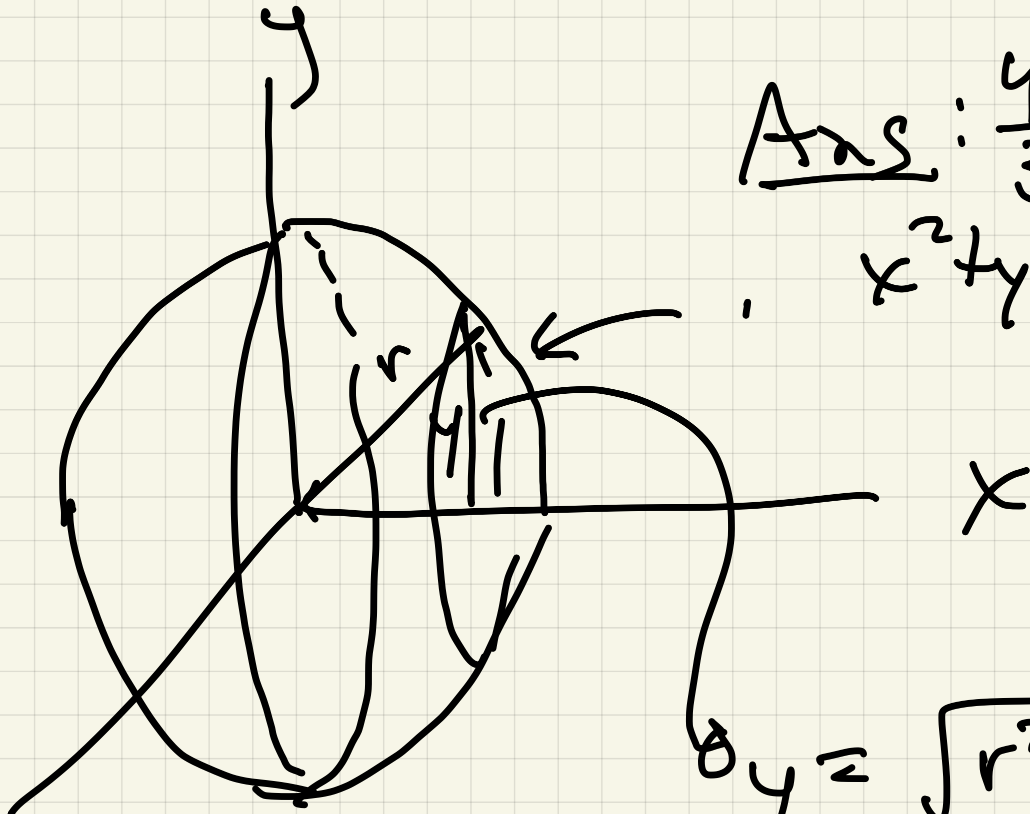
$$du = -\frac{1}{2}\,dx$$

$$-2\,du = dx$$

$$\int_{u=1}^{u=0} \cancel{\frac{1}{2}}\, u^2 \left(-\cancel{2}\, du\right)$$

$$\int_0^1 u^2\,du = \frac{1}{3}$$

__Ex 3__ Find volume of a sphere of radius r



Ans: $\frac{4}{3}\pi r^3$

$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

so radius of cross section circle at position x is $y=\sqrt{r^2-x^2}$

so Area of circle $\pi y^2 =$

$\pi(r^2-x^2)$

And $V=\int_{-r}^{r}\pi(r^2-x^2)\,dx =$

$\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_{-r}^{r} =$

$$\pi\left(\underbrace{r^3-\frac{r^3}{3}}\right)-\pi\left(-r^3+\frac{r^3}{3}\right)$$

$$\pi\left(\tfrac{2}{3}r^3\right)-\pi\left(-\tfrac{2}{3}r^3\right)=$$

$$\frac{4\pi}{3}r^3 \checkmark$$

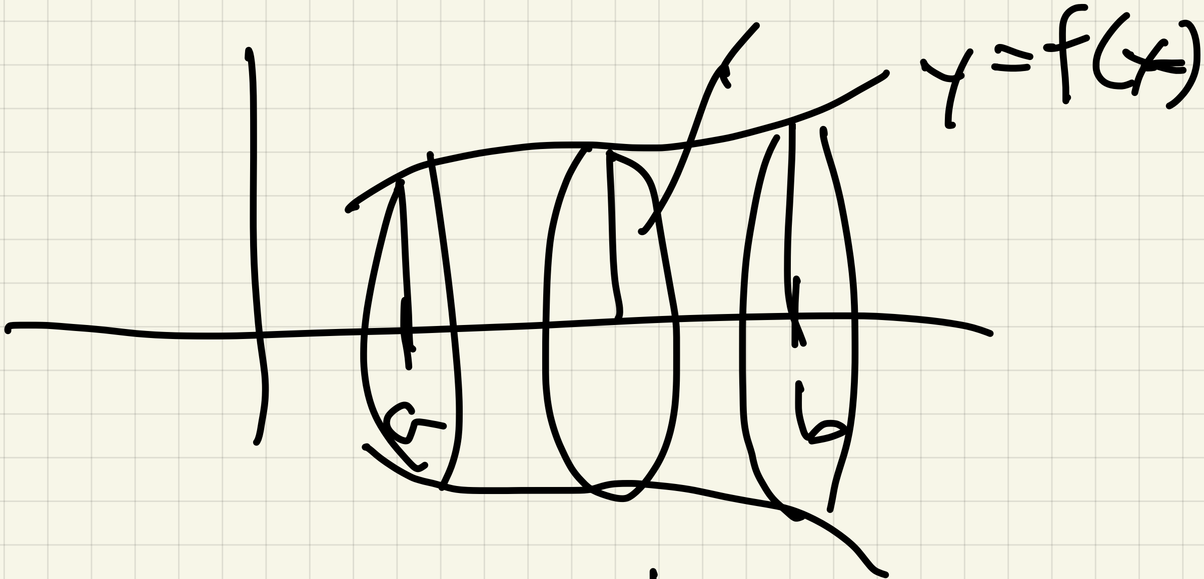
In general, if cross sections are circles, then

$$V=\int_a^b \pi\left(R(x)^2\right)dx$$

$R(x)$ = radius of cross section circle

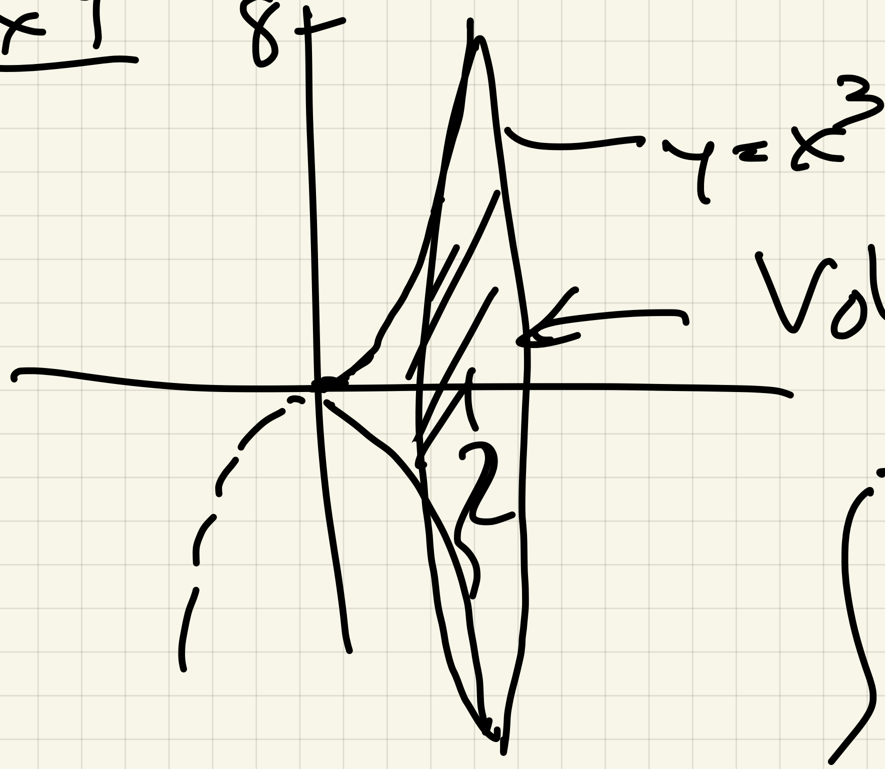
"Disk Method"

$R(x)=f(x)$



so $V = \int_a^b \pi (f(x))^2 \, dx$

Ex 4



Volume inside

"

$$\int_0^2 \pi R(x)^2 \, dx$$

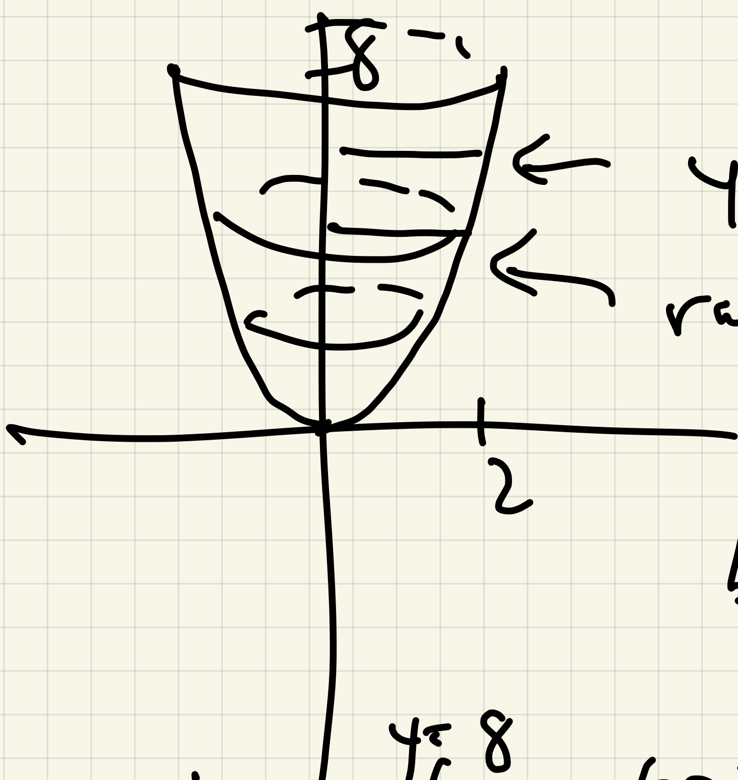
$y = x^3$

$0 \le x \le 2$

$$\int_0^2 \pi (x^3)^2 \, dx =$$

$$\int_0^2 \pi x^6\,dx = \frac{\pi x^7}{7}\Big|_0^2 = \frac{128\pi}{7}$$

Ex5 what is volume results if we revolve $y=x^3$ $0\le x\le 2$ about $\underline{y}$-axis?



$\leftarrow$ $y=x^3$

$\leftarrow$ radius of circle at height y??

Ans $x=\sqrt[3]{y}$

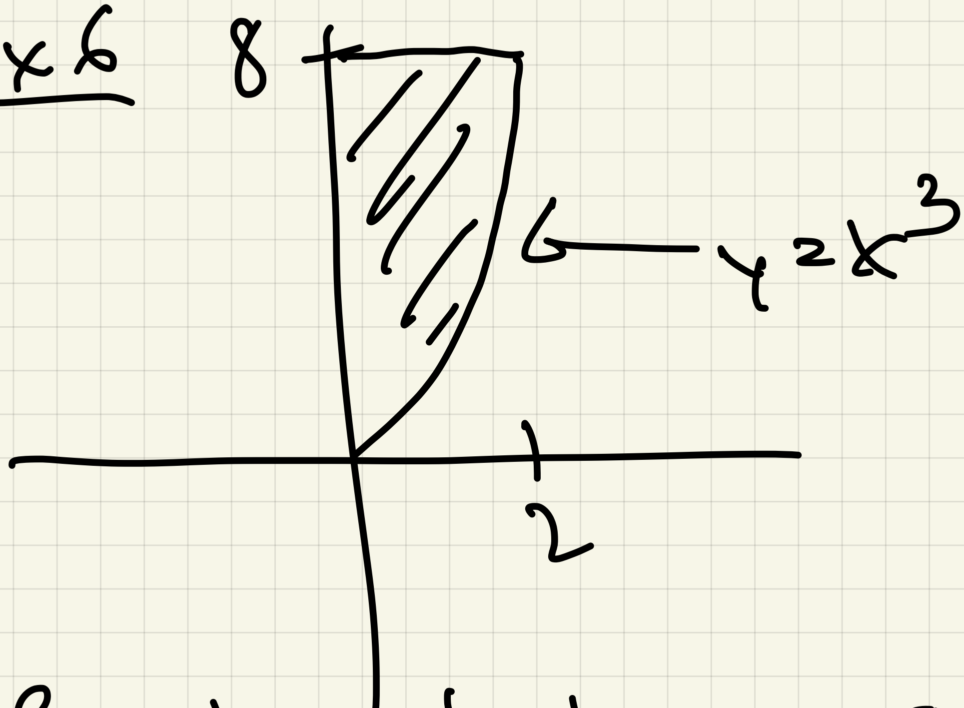
$$V=\int_0^{y=8} \pi\underbrace{\left(\sqrt[3]{y}\right)^2}\,dy = \int_0^8 \pi y^{2/3}\,dy =$$

$$\pi\,\frac{3}{5}y^{5/3}\Big|_0^8 =$$

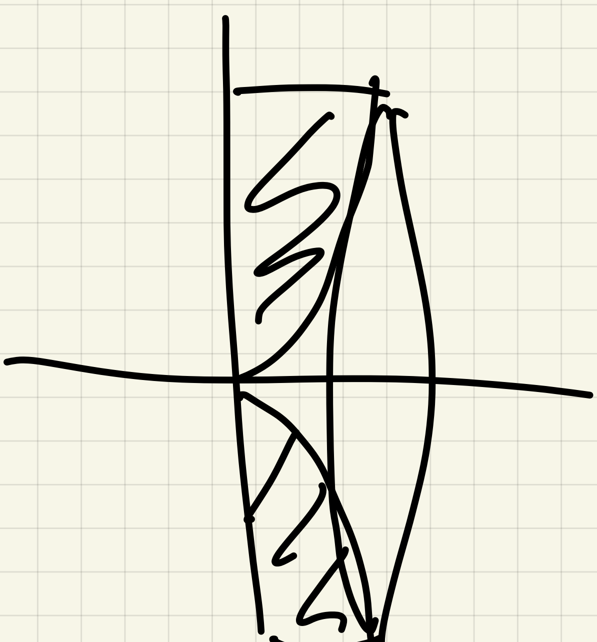
$$\frac{3\pi}{5} 8^{5/3} = \frac{3}{5}\pi \left(\underbrace{8^{1/3}}\right)^5$$

$$\frac{3}{5}\pi \cdot 2^5 = \frac{3 \cdot 32\pi}{5} = \frac{96\pi}{5}$$

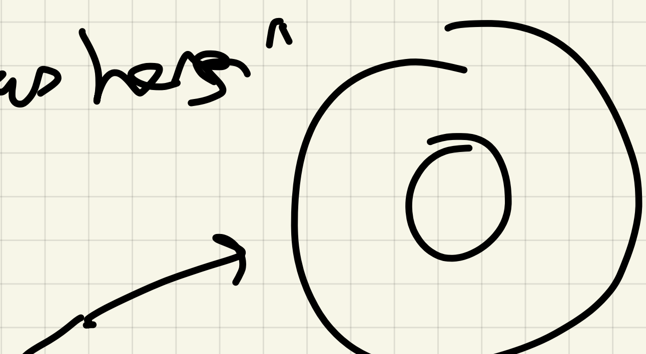
Ex 6



Revolve about x-axis,
find volume



cross-sections
are
"washers"



so

Area

$\pi\left(\text{outer radius}\right)^2 - \pi\left(\text{in rad.}\right)^2$

in our situation

$$\int_0^2 \pi\left(8^2 - \left(x^3\right)^2\right)dx$$

$$\parallel$$

$$\int_0^2 \pi\, 8^2\, dx - \int_0^2 \pi\left(x^3\right)^2 dx$$

$$\pi\int_0^2 64\, dx$$

$$= \pi \cdot 64 \cdot 2 =$$

$$128\pi - \frac{128\pi}{7}$$

$$\parallel$$

$$\frac{768\pi}{7}$$