Taylor Polynomials and Taylor Series **May, 2026**
Calculus II **Prof. Nollet**

1. **Power series.** These are functions of the form $f(x) = \sum_{k=0}^{\infty} a_k\ (x-c)^k$, c is the center and a_k are the coefficients. Using the ratio test one finds the radius of convergence R such that $f(x)$ converges for $|x-c| < R$ and diverges for $|x-c| > R$. Note that this test FAILS at the endpoints $x = c \pm R$ must be tested by other methods.

   You can compute $f'(x)$ and $\int f(x)\ dx$ term by term using the power rule from Calc I. The radius of convergence remains the same, but convergence at the endpoints may change.

   For each power series function below, find the interval of convergence, the radius of convergence, the derivative and the integral.

   (a) $\sum_{k=1}^{\infty} \frac{k}{12^k}(x-3)^k$ (b) $\sum_{k=1}^{\infty} \frac{k^2+2}{k!}x^k$ (c) $\sum_{k=1}^{\infty} \frac{7^k}{k2^k}(x-1)^k$ (d) $\sum_{k=1}^{\infty} k5^kx^k$

2. **Taylor Polynomial and estimates.** Given a function $f(x)$ defined at $x = c$, the nth degree Taylor polynomial is $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ centered at $x = c$. Compute the following Taylor polynomials.

   (a) $f(x) = \sin 2x, c = 0, n = 4$
   (b) $f(x) = \sqrt{x}, c = 25, n = 2$
   (c) $f(x) = \ln(x+1), c = 0, n = 3$
   (d) $f(x) = e^x, c = 0, n = 4$

   Often $f(x) \approx P_n(x)$ for x near c. If $f(x) = P_n(x)+R_n(x)$, then Taylor's formula says that $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ for some z between c and x. Using answers (a)-(d), use the Taylor polynomial to estimate the following and use Taylor's theorem to bound error in the estimates.

   (a) $\sin 2 \approx P_4(1)$ (b) $\sqrt{26} \approx P_2(1)$ (c) $\ln 1.3 \approx P_3(.3)$ (d) $\sqrt{e} \approx P_4(1/2)$

3. **Taylor series by definition.** The Taylor *series* for $f(x)$ at $x = c$ is $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$. It is called Maclaurin series when $c = 0$. Compute the Taylor series for these functions (note, you can use your start on parts (a), (c), (d) from above):

   (a) $f(x) = \sin 2x, c = 0;$

   (b) $f(x) = x^4, c = 2;$

   (c) $f(x) = \ln(x+1), c = 0;$

   (d) $f(x) = e^x, c = 0;$

4. **Taylor series by operations** If $f(x)$ is a function and $R_n(x)$ is the Taylor remainder, then $f(x)$ is **equal** to its Taylor series exactly when $\lim\limits_{n\to\infty} R_n(x) = 0$. In particular, we have the known Maclaurin series

   (a) $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$

   (b) $\cos x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k\, x^{2k}}{(2k)!}$

   (c) $\sin x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!}$

   (d) $\frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots \quad -1 < x < 1$

   Starting from these it can be MUCH easier to find Taylor series for related functions by using *operations* such as addition, substitution, differentiation, integration or multiplication. Starting with the series above, find the Taylor series for the following functions:

   (a) $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$

   (b) $\cos x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k\, x^{2k}}{(2k)!}$

   (c) $\sin x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!}$

(d) $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots \quad -1 < x < 1$

Use these to find Maclaurin series for the following functions:

(a) $f(x) = \sin 2x$. Did you get the same answer as 3(a)?

(b) $f(x) = \ln(1+x)$. Start with (d), replace x with $-x$, then integral both sides. Did you get the same answer as 3(c)?

(c) $f(x) = x^2 e^{7x}$

(d) $f(x) = 3x^2 \cos x^3$ [hint: (i) substitute x^3 for x in $\cos x$ and multiply by $3x^2$ **OR** (ii) start with $\sin x$, substitute x^3 for x and take the derivative].

(e) $f(x) = e^{x^5}$

(f) $f(x) = \dfrac{3}{(1-x)^2}$ [differentiate the geometric series].

5. **Applications.** Using Taylor series, there are many calculations that become possible or easier. See problems from Homework 21.