

10/22/Calc2

Exam 2 → Tuesday 10/27

Review sheet

all HW/quiz back Monday

Quiz 12:

$$\textcircled{1} \lim_{x \rightarrow 0} \frac{e^{4x} - 4x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{4e^{4x} - 4}{6x} =$$

$$\lim_{x \rightarrow 0} \frac{16e^{4x}}{6} = \frac{16}{6} = \frac{8}{3}$$

$$\textcircled{2} y = \lim_{x \rightarrow 0^+} (1-3x)^{6/x} \Rightarrow \ln y = \lim_{x \rightarrow 0^+} \ln(1-3x)^{6/x}$$

$$= \lim_{x \rightarrow 0^+} \frac{6 \ln(1-3x)}{x} = \lim_{x \rightarrow 0^+} \frac{6 \frac{1}{(1-3x)} (-3)}{1} =$$

$$-18, \text{ so } y = e^{-18}$$

Last time

Sequences $\{a_n\}_{n \geq 1}$

Convergence

$$\lim a_n = L$$

or

$$q_n \rightarrow L$$

Calc limits apply:

Ex) $a_n = (1 - 3/n)^{6n}, n \geq 1$

$$a_1 = (-2)^6 = 64 \quad a_2 = (1 - 3/2)^{12} = \frac{1}{4096} \quad a_3 = 0^{18} = 0$$

$$\lim_{n \rightarrow \infty} (1 - 3/n)^{6n} = \lim_{x \rightarrow \infty} (1 - 3/x)^{6x} =$$

$$u = \frac{1}{x}, \text{ so } u \rightarrow 0^+ \text{ as } x \rightarrow \infty,$$

$$\lim_{u \rightarrow 0^+} (1 - 3u)^{6/u} = e^{-18}$$

$$(b) \lim_{n \rightarrow \infty} \left(\frac{1}{e} \right)^n + \frac{n^2 - 1}{6 - 3n^2} + \arctan n$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{e} \right)^x + \frac{x^2 - 1}{6 - 3x^2} + \arctan x =$$

$\downarrow 0 \quad \downarrow -\frac{1}{3} \quad \downarrow \pi/2$

$$\pi/2 - 1/3 = \frac{3\pi - 2}{6}$$

Ex 2 Not always:

(a) $\lim_{n \rightarrow \infty} \sin(n\pi) = 0$ b/c

$\sin(n\pi) = 0$ all n , but

$\lim_{x \rightarrow \infty} \sin(\pi x)$ DNE b/c graph oscillates from -1 to 1



(b) $\lim_{n \rightarrow \infty} \cos n\pi = ?$

n	1	2	3	4	5	...
$\cos(n\pi)$	-1	1	-1	1	-1	...

so $\cos n\pi = (-1)^n$ and

$\lim_{n \rightarrow \infty} \cos n\pi$ diverges odd terms = -1
even terms = 1

$\cos(n\pi)$ doesn't approach
one limiting value L .

$\cos(n\pi)$ oscillates from -1 to 1

(c) Oscillation $\nrightarrow$ divergence!

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = ?$$

n	1	2	3	4	5
$\frac{(-1)^n}{n}$	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$

terms approach 0 as $n \rightarrow \infty$,
though oscillation is seen.

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Remark One way to see this is
notice

$$\begin{array}{ccc} -\frac{1}{n} & \leq & \frac{(-1)^n}{n} \leq \frac{1}{n} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Since $-\frac{1}{n}, \frac{1}{n} \rightarrow 0$, $\frac{(-1)^n}{n} \rightarrow 0$ also

since limit is squeezed in:

Sandwich Thm (in book):

If $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = L$,

Then $\lim_{n \rightarrow \infty} b_n = L$ also.

Concepts : $\{a_n\}_{n=1}^{\infty}$ sequence

① $\{a_n\}$ is monotonic if

(a) $a_1 \leq a_2 \leq a_3 \leq \dots$ (non-decrease)
or

(b) $a_1 \geq a_2 \geq a_3 \geq \dots$ (non-increase)

② $\{a_n\}$ is bounded above if

there exists M : $a_n \leq M$ all n

Similar definition for bounded below

③ $\{a_n\}$ is bounded if both
bounded above and below.

Thm If $\{a_n\}$ is monotonic, then

$\{a_n\}$ converges $\iff \{a_n\}$ is bounded

Ex 3 (a) $a_n = \frac{n}{n+1}$ is monotonically increasing

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \dots$$

Can show with calc 1:

$$f(x) = \frac{x}{x+1}, \quad f'(x) = \frac{+1}{(x+1)^2} > 0 \Rightarrow$$

f increasing

a_n bounded above $\frac{1}{2} < a_n < 1$

a_n converges to 1

(b) $b_n = -(2^n), n \geq 0$

$$-1 \quad -2 \quad -4 \quad -8 \quad -16, \dots$$

b_n is decreasing, hence monotonic. Since $2^n \rightarrow +\infty$,

b_n is not bounded below,

$\therefore b_n$ diverges.

$$(c) \quad c_n = \frac{n^{60}}{e^n}, \quad n \geq 0$$

Is c_n monotone? convergent?
Bounded?

A few terms?

$$c_0 = 0$$

$$c_1 = \frac{1}{e} \approx .368$$

$$c_2 = \frac{2^{60}}{e^2} \approx 1.56 \times 10^{17}$$

$$c_3 = \frac{3^{60}}{e^3} \approx 2.11 \times 10^{27}$$

looks to be increasing rapidly

bounded below by 0

monotone?

Calc: $f(x) = \frac{x^{60}}{e^x}$;

$$f'(x) = \frac{e^x \cdot 60x^{59} - x^{60}e^x}{(e^x)^2} = \frac{x^{59}(60-x)}{e^x}$$

so

$$f'(x) > 0 \quad x < 60$$

$$f'(x) < 0 \quad x > 60$$

$\therefore$ c_n not monotone :

increasing for $n \leq 60$, decreases $n \geq 60$

Also c_n bounded above by
 $c_{60} \approx 4.28 \times 10^{80}$

Finally, $\lim_{x \rightarrow \infty} \frac{x^{60}}{e^x} \underset{\infty}{=} \lim_{x \rightarrow \infty} \frac{60x^{59}}{e^x} \ll \dots$

$$= \lim_{x \rightarrow \infty} \frac{60 \cdot 59 \cdot 58 \cdot \dots \cdot 1}{e^x} = \lim_{x \rightarrow \infty} \frac{60!}{e^x} = 0$$

so $\lim_{n \rightarrow \infty} c_n = 0$

Remark: Theorem doesn't apply directly, but can apply it to the sequence $\{c_n\}_{n \geq 60}$,

Which is ~~wasteful~~ decreasing.

§9.2 The rest of Chapter 9
concerns this:

Defn If $\{a_n\}_{n \geq 1}$ is a sequence, then

$$a_1 + a_2 + \dots = \sum_{k=1}^{\infty} a_k$$

is an infinite series with
terms a_k

Adding infinitely many numbers
doesn't make sense, so we interpret
using limits?

Set $S_n = a_1 + \dots + a_n = \sum_{k=1}^n a_k =$
 n^{th} partial sum

Then define

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n$$

so $\sum_{k=1}^{\infty} a_k = L$ if $\lim S_n = L$

$\sum_{k=1}^{\infty} a_k$ diverges if $\lim S_n$ ~~diverges~~

Ex1 If $a_k = 1$ all k ,

$$\sum a_k = \sum 1 = 1 + 1 + 1 + \dots$$

$$S_n = n, \quad \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} n = \infty$$

diverges, so $\sum_{k=1}^{\infty} a_k$ diverges

Ex2 $a_k = \frac{5}{k(k+1)}$

$$\frac{5}{2} + \frac{5}{6} + \frac{5}{12} + \frac{5}{20} + \dots$$

$$S_1 = \frac{5}{2} = 5\left(\frac{1}{2}\right)$$

$$S_2 = \frac{5}{2} + \frac{5}{6} = \frac{20}{6} = \frac{10}{3} = 5\left(\frac{2}{3}\right)$$

$$S_3 = \frac{5}{2} + \frac{5}{6} + \frac{5}{12} = \frac{10}{3} + \frac{5}{12} = \frac{45}{12} = 5\left(\frac{3}{4}\right)$$

Looks like $S_n = 5\left(\frac{n}{n+1}\right)!$