11/9/ Calc 2

Quiz 16
avg 80

1. $\sum_{n=1}^{\infty} \frac{10}{5^n} = 2 + \frac{2}{5} + \frac{2}{25}$

geom $r = \frac{1}{5} < 1$

$\therefore$ converges

2. $\sum_{n=1}^{\infty} \frac{1}{n^5}$

p-series, $p = 5 > 1$

$\therefore$ converges

3. $\sum \frac{3n^3}{8n^3+1}$

$\lim_{n\to\infty} \frac{3n^3}{8n^3+1} = \frac{3}{8} \neq 0$

$\therefore$ diverges by nth term test

4. $\sum_{n=1}^{\infty} \frac{8n}{n^2+1}$

$f(x) = \frac{8x}{x^2+1}$

Int test:

pos ✓, cont ✓
dec ✓

$$f'(x) = \frac{(x^2+1)\,8 - 8x(2x)}{(x^2+1)^2} =$$

$$\frac{8(1-x^2)}{(x^2+1)^2} < 0 \Rightarrow \text{decr}$$

$$\int_1^\infty \frac{8x}{x^2+1} = \lim_{b\to\infty} 4\ln(x^2+1)\Big|_1^b =$$

$$\lim_{b\to\infty} 4\ln(b^2+1) - 4\ln 2 \to \infty$$

$\to \infty$

$\therefore$ div by int test.

Last time

n$^{\text{th}}$ term test

geom

telescoping

→ Integral test

→ p-series

→ DCT

→ LCT

Fact: $\sum_{n=1}^{\infty} |a_n|$ conv $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv

Ratio test $r = \lim \left| \frac{a_{n+1}}{a_n} \right|$

$r < 1 \Rightarrow \sum a_n$ abs conv

$r > 1 \Rightarrow$ diverges.

$r = 1 \Rightarrow$ test fails

Rmk: Ratio works well for factorials & exponentials

Ex1 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$r = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} =$$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} =$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$=$$

$$L = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\ln L = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)^n =$$

$$\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) =$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)}\cdot\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= 1$$

$$\therefore L = e^1 = e > 1$$

$\therefore$ diverges by ~~ratio~~ root test

Esempio: $\frac{n^n}{n!} = \frac{n\cdot n\cdot n \cdots n}{n(n-1)(n-2)\cdots(1)} \geq$

$$\underbrace{1\cdot 1\cdot 1\cdots 1}_{n} = 1$$

$\sum_{n=1}^{\infty} 1$ div geom series $(r=1)$

DCT $\Rightarrow$ $\sum \frac{n^n}{n!}$ diverges

Root test Set $r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

$r<1 \Rightarrow \sum a_n$ conv absolutely

$r>1 \Rightarrow$ diverges

$r=1 \Rightarrow$ test fails

Ex2 (a) $\sum_{n=1}^{\infty} \frac{4}{3^n}$ (geom ser $r=\frac{1}{3}<1$ conv.)

Root test: $r = \lim_{n\to\infty} \sqrt[n]{\frac{4}{3^n}} =$

$\lim_{n\to\infty} \frac{\sqrt[n]{4}}{3} = \lim_{n\to\infty} \frac{4}{3} = \frac{1}{3} < 1$ ($\sqrt[n]{4} \to 0$)

$\therefore$ conv absolutely by ratio test.

(b) $\sum_{n=1}^{\infty} \left(\frac{8}{n}\right)^n$ $\quad r = \lim_{n\to\infty} \sqrt[n]{\left(\frac{8}{n}\right)^n}$

$= \lim_{n\to\infty} \frac{8}{n} = 0 < 1$

$\therefore$ <u>conv. abs.</u>

(c) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$

$r = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1$ [Ex]

$\therefore$ <u>diverges</u> by <u>root test</u>

<u>Easier</u>: $\left(\frac{n+1}{n}\right)^{n^2} \geq 1^{n^2} = 1$

$\sum 1$ <u>diverges</u>, so

geom $r=1$ DCT $\Rightarrow$ $\sum \left(\frac{n+1}{n}\right)^{n^2}$ <u>div</u>

(d) $\sum \frac{1}{n^2}$ (p-series, $p=2>1$, conv.)

Root test: $r = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}}$

$L = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}}$ "0^0"

$\ln L = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{n^2}\right) =$

$\lim_{n\to\infty} \frac{\ln\left(\frac{1}{n^2}\right)}{n}$ LH $=$

$\lim_{n\to\infty} \frac{1}{\left(\frac{1}{n^2}\right)} \cdot \frac{-2}{n^3} = \lim_{n\to\infty} \frac{-2}{n} = 0$

$\therefore L = e^0 = 1$

9.6 A series $\sum b_n$ is alternating if signs of terms

alternate $+/-1+/-/+--$.

Ex) $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \frac{3}{625} - \ldots$

(geom conv to $\frac{5}{2}$)

Note If $\sum b_n$ alternates, can pull out sign $(-1)^n$ or $(-1)^{n+1}$ to write a

$\sum b_n = \sum (-1)^n a_n$ $\quad a_n \geq 0$

$\sum (-1)^{n+1} a_n$

AST Alternating Series test:

① $\lim a_n = 0$
② a_n decr

$\Rightarrow \sum (-1)^n a_n$ converges.