11/5 / Calc 2 Quiz 15

(1) $\sum_{k=1}^{\infty} 20\left(-\frac{4}{5}\right)^k$

Col

$-16 + \frac{64}{5} - \frac{256}{25} + \cdots$

$-1 < r = -\frac{4}{5} < 1 \Rightarrow$ convergent

$$\frac{a}{1-r} = \frac{-16}{1-\left(-\frac{4}{5}\right)} = \frac{-16}{\frac{9}{5}} = \frac{-80}{9}$$

(b) $\sum_{k=1}^{\infty} \left(\frac{k}{10k+100}\right)^2$

$\lim_{k\to\infty} \left(\frac{k}{10k+100}\right)^2 =$

$\left(\lim_{k\to\infty} \frac{k}{10k+100}\right)^2 = \left(\frac{1}{10}\right)^2 = \frac{1}{100}$

$\neq 0$

$\Leftarrow$ nth term test

divergent

(2) $\sum_{k=1}^{\infty} \frac{6}{(3k-2)(3k+1)} = \underbrace{\frac{6}{4}}_{a_1} + \frac{6}{28} + \frac{6}{70} + \frac{6}{135}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad a_1 \quad a_2 \quad a_3$

(a)

$S_1 = \frac{6}{4}$

$S_2 = \frac{6}{4} + \frac{6}{28} = \frac{48}{28} = \frac{12}{7}$

$S_3 = \underbrace{\frac{6}{4} + \frac{6}{28}} + \frac{6}{70} = \frac{12}{7} + \frac{6}{70} = \frac{126}{70}$

$\frac{9}{5}$

(b)

$S_n = a_1 + a_2 + \cdots + a_n$

$= \left(\frac{2}{1} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{7}\right) + \cdots + \left(\frac{2}{3n-2} - \frac{2}{3n+1}\right)$

$= 2 - \underbrace{\frac{2}{3n+1}} \to \underbrace{2}$

$\quad\;\; \| \qquad\qquad\quad \text{Sum}$

S_n

Last time:

DCT $0 \le a_n \le b_n$

① $\sum b_n$ conv $\Rightarrow \sum a_n$ conv.

② $\sum a_n$ div $\Rightarrow \sum b_n$ div.

LCT $0 < a_n, b_n$

$$r = \lim_{n\to\infty} \frac{a_n}{b_n}$$

① $0 < r < \infty \Rightarrow$

$\sum a_n, \sum b_n$ both converge

or

both diverge

② $r = 0$, $\sum b_n$ conv $\Rightarrow \sum a_n$ conv.

③ $r = \infty$, $\sum b_n$ div $\Rightarrow \sum a_n$ div

Ex 1 (a) $\sum_{n=2}^{\infty} \frac{12}{\sqrt[8]{n^9-17}} \sim \frac{1}{\sqrt[8]{n^9}}$

$$\lim_{n\to\infty} \frac{\frac{12}{\sqrt[8]{n^9-17}}}{\frac{1}{\sqrt[8]{n^9}}} = \lim_{n\to\infty} 12\sqrt[8]{\frac{n^9}{n^9-17}} =$$

$$\lim_{n\to\infty} 12\sqrt[8]{\frac{1}{1-17/n^9}} = 12 \neq 0, \infty$$

$\sum \frac{1}{\sqrt[8]{n^9}}$ is conv p-series

$p = 9/8 > 1$

$\therefore$ LCT $\Rightarrow \sum \frac{12}{\sqrt[8]{n^9-17}}$ conv.

(b) $\sum_{n=2}^{\infty} \frac{\sqrt[2]{3n^3+1}}{\sqrt[7]{7n^{12}-2}} \sim \frac{n^{3/2}}{n^{12/7}} =$

$$\frac{1}{n^{12/7 - 3/2}} = \frac{1}{n^{3/14}} \quad (\text{div.})$$

$$\lim \frac{\sqrt[2]{3n^3+1}}{\sqrt[7]{7n^{12}-2}} = \lim \sqrt[2]{\frac{3n^3+1}{n^3}} \cdot \frac{n^{3/2}}{n^{12/7}} \sqrt[7]{\frac{n^{12}}{7n^{12}-2}}$$

$$= \lim_{n\to\infty} \sqrt[2]{\frac{3+1/n^3}{1}} \cdot \sqrt[7]{\frac{1}{7-2/n^{12}}}$$

$$= \frac{\sqrt{3}}{\sqrt[7]{7}} \neq 0, \infty$$

$\sum \frac{1}{n^{3/14}}$ div p-Srie $p = 3/14 < 1$

$\therefore \sum \frac{\sqrt{3n^3+1}}{\sqrt[7]{7n^{12}-2}}$ div by LCT,

Rmk: LCT, DCT, Int test, work for series $\sum a_k$ with $a_k \geq 0$. Often this is good enough:

Def: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges

(ACT) Absolute convergence theorem

$\sum a_n$ abs. conv, $\Rightarrow \sum a_n$ conv,

Ex) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

conv p-series $(p = 3 > 1)$

ACT $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ conv.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$n!_1 = \begin{cases} 1 & n = 0 \\ n(n-1)(n-2)\cdots(1) & n > 0 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!_1}$$

For $n \geq 2$, $\frac{1}{n(n-1)\cdots(1)} \leq \frac{1}{n(n-1)} \leq \frac{1}{(n-1)^2}$

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

conv p-series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \text{ converges}$$

$\therefore$ AS $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

Ratio Test: $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$\sum a_n$

1. $r < 1 \Rightarrow \sum a_n$ conv. absolutely
2. $r > 1 \Rightarrow \sum a_n$ diverges
3. $r = 1 \Rightarrow$ Test fails

Ex 1(b) revisited: Apply ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} =$$

$$\lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

$\therefore$ series absolutely convergent

Ex 2: $\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$ [Know conv. geom series]

(a)

Ratio test: $\lim_{n\to\infty}\left|\frac{\left(\frac{-2}{3}\right)^{n+1}}{\left(\frac{-2}{3}\right)^n}\right| =$

$\lim_{n\to\infty}\frac{2}{3} = \frac{2}{3} < 1$

(b) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$r = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \right| =$

$\lim_{n\to\infty} \frac{n+1}{100} = +\infty > 1$

diverges.

(c) $\sum_{n=1}^{\infty} \frac{n^{60}}{e^n}$

$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{60}}{e^{n+1}}}{\frac{n^{60}}{e^n}} \right| =$

$$\lim_{n\to\infty} \frac{1}{e}\left(\frac{n+1}{n}\right)^{60} = \frac{1}{e} < 1$$

$\therefore$ abs convergent.

(b) $\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{n^2}$

$$r = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{(n+1)^2} \cdot \frac{\sqrt{n-1}}{n^2} \right| =$$

$$\lim_{n\to\infty} \sqrt{\frac{n}{n-1}} \cdot \left(\frac{n}{n+1}\right)^2 = 1$$

Ratio test fails:

$$\frac{\sqrt{n-1}}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}}$ conv, p-ser $\Rightarrow$

$\therefore \sum \frac{\sqrt{n-1}}{n^2}$ converges.