11/2 | Calc 2

9.1 $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$

$= \lim_{n \to \infty} S_n$

$S_n = \sum_{k=1}^{n} a_k = a_1 + \dots + a_n$

Telescoping series

Geometric series

$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$

converges to $\frac{a}{1-r}$ if $-1 < r < 1$

diverges otherwise

n^{th} term test

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges

Integral test:

If $f(x)$ is positive, continuous, decreasing for $x \geq 1$
(also ok if $x \geq N$, any N) then

$$\sum_{n=1}^{\infty} f(n) \text{ conv} \iff \int_1^{\infty} f(x)\,dx \text{ conv}$$

Consequence:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1$$

↖ p-series

(p-series test)

Ex 1 Test for convergence

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series $p = \frac{1}{2} \leq 1$

(b) $\sum_{n=5}^{\infty} \frac{1}{n^3}$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series

$\Rightarrow \sum_{n=5}^{\infty} \frac{1}{n^3}$ also converges

(c) $\sum_{n=1}^{\infty} \ln n = 0 + \ln 2 + \ln 3 + \cdots$

diverges b/c $\lim_{n\to\infty} \ln n = \infty \neq 0$

(d) $\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{3n-1}$

Int test:

$\int_1^{\infty} \frac{1}{3x-1}$

note: $\frac{1}{3x-1}$ continuous, pos ✓ for $x \geq 1$ dec ✓

$\lim_{b\to\infty} \int_1^b \frac{1}{3x-1}\,dx =$

$$\lim_{b\to\infty} \frac{1}{3}\ln|3x-1|\Big|_1^b =$$

$$\lim_{b\to\infty} \frac{1}{3}\ln|3b-1| - \frac{1}{3}\ln 2 = +\infty$$

Int test

$\therefore$ diverges

$$\sum \frac{1}{3n-1} \text{ diverges}$$

(e) $$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

$f(x) = \frac{\ln x}{x^3}$ cont, pos

deer?

$$f'(x) = \frac{x^3\frac{1}{x} - \ln x \cdot 3x^2}{x^6}$$

$$= \frac{x^2(1-3\ln x)}{x^6}$$

$$= \frac{1-3\ln x}{x^4} < 0 \text{ for } x > e^{1/3} \approx 1.4$$

ok for $x \geq 2$

$$\int_2^\infty \frac{\ln x}{x^3}\,dx = -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^3}$$

$u = \ln x \quad dv = \frac{1}{x^3}\,dx$

$du = \frac{1}{x}\,dx \quad v = -\frac{1}{2x^2}$

$$-\frac{\ln x}{2x^2} - \frac{1}{4x^2}$$

$$\int_2^\infty \frac{\ln x}{x^3} = \lim_{b\to\infty} -\frac{\ln x}{2x^2} - \frac{1}{4x^2}\Big|_2^b$$

$\frac{1}{4x^2} \to 0$

$$\lim_{b\to\infty} -\frac{\ln b}{2b^2} \overset{LH}{=} \lim_{b\to\infty} \frac{-\frac{1}{b}}{4b} =$$

$$\lim_{b\to 0} \frac{-1}{4b^2} = 0$$

So $\int_2^\infty$ converges

$\therefore \sum \frac{\ln n}{n^3}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ n^{th} term fails

can use telescoping series

can integral test

Easier way: $\frac{1}{n(n+1)} > 0 \Rightarrow$ partial sums S_n increase for convergence, enough to show bounded above.

BVT: $\frac{1}{n(n+1)} \leq \frac{1}{n^2}$

particle $S_n \leq t_n$ bdd

t_n bdd $\Rightarrow$ S_n bdd

and $\sum \frac{1}{n^2}$ conv ($p=2$ p-Ser)

s_0 t_n and $\Rightarrow$ t_n bdd $\Rightarrow$

s_n bdd $\Rightarrow$ $s_n \to$ converges