11/10/Calc2 $\sum a_n$ n^{th} term test

Series tests

geom
telescoping
integral test
p-series
DCT
$\Rightarrow$ LCT
Ratio
Root

ACT

$\sum |a_n|$ conv $\Rightarrow$ $\sum a_n$ conv.

AST: For $a_n \geq 0$

① $\lim a_n = 0$
② a_n decreases
$\Rightarrow \sum (-1)^n a_n$ conv
(or $\sum (-1)^{n+1} a_n$ conv.)

Ex1 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

$\lim \frac{1}{n} = 0$ ✓

$\frac{1}{n}$ decreases: $\frac{1}{n} > \frac{1}{n+1}$ ✓ $\Rightarrow$ converges (to $\ln 2$)

Rmk: $\sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$ is

divergent p-series, so ACT doesn't apply.

Defn: $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Only 3 possibilities:

1. abs convergence
2. conditional converge
3. divergence

Ex2 Determine which for. following

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$

AST $\Rightarrow$ converges, BUT check for absolute convergence.:

Ratio test: $\lim \left| \frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}} \right| =$

$\lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \Rightarrow$ abs converg.

$\therefore$ abs. convergent :

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

check abs conv: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Integral test:

$f(x) = \frac{1}{x \ln x}$

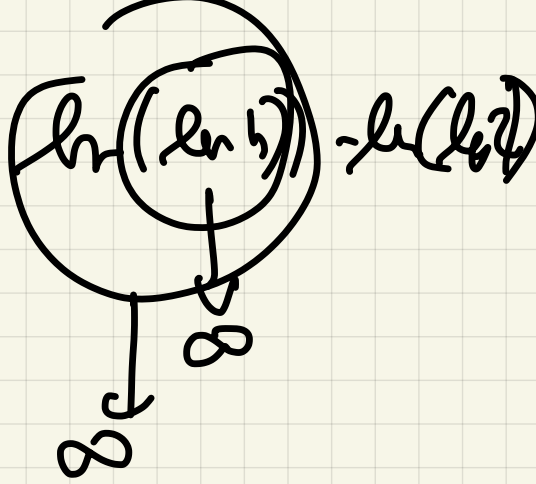
pos ✓
cont ✓
decr ✓

$\int_2^{\infty} \frac{1}{x \ln x} =$

$\int \frac{du}{u} = \ln u$
$= \ln(\ln x)$

$u = \ln x$
$du = \frac{1}{x} dx$

$$\lim_{b\to\infty}\int_2^b \frac{1}{x\ln x} = \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2)$$



diverges:

AST: $\lim \frac{1}{x\ln x} = 0$

$\frac{1}{n\ln n}$ decr

$\Rightarrow$ conv

$\therefore$ cond. conv.

(c) $\sum_{n=1}^{\infty}(-1)^{n+1} = 1-1+1-1+1-\cdots$

AST does not apply!

n^{th} term test $\Rightarrow$ diverges.

(d) $\sum \frac{(-1)^n n^{60}}{e^n}$

Test for abs conv:

$\sum \frac{n^{60}}{e^n}$ ratio test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{60}}{e^{n+1}}}{\frac{n^{60}}{e^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{60} \frac{1}{e}$$

$$= 1 \cdot \frac{1}{e} < 1$$

$\therefore$ abs. conv.

§9.7 We've seen that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad -1 < r < 1$$

So if we think r as variable x, a = 1

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$-1 < x < 1$ $(-1, 1)$

well understood function

infinite series

Defn: If x is a variable, a **power series** has form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

A **power series centered at $x=c$** has form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + \dots$$

Defn: The **interval of convergence** is $\{x;\ f(x) \text{ converges}\}$

Ex1: $f(x) = \sum_{n=0}^{\infty} \frac{1}{5^n} (x-4)^n$ I.O.C.

find I.O.C.

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{1}{5^{n+1}}(x-4)^{n+1}}{\frac{1}{5^n}(x-4)^n} \right| =$

$\lim_{n\to\infty} \frac{|x-4|}{5} = \frac{|x-4|}{5}$

conv. if $\frac{|x-4|}{5} < 1$

div. if $\frac{|x-4|}{5} > 1$

$\frac{|x-4|}{5} < 1 \Rightarrow \boxed{|x-4| < 5} \Leftrightarrow$

$-5 < x-4 < 5 \quad +4$

$-1 < x < 9$ conv

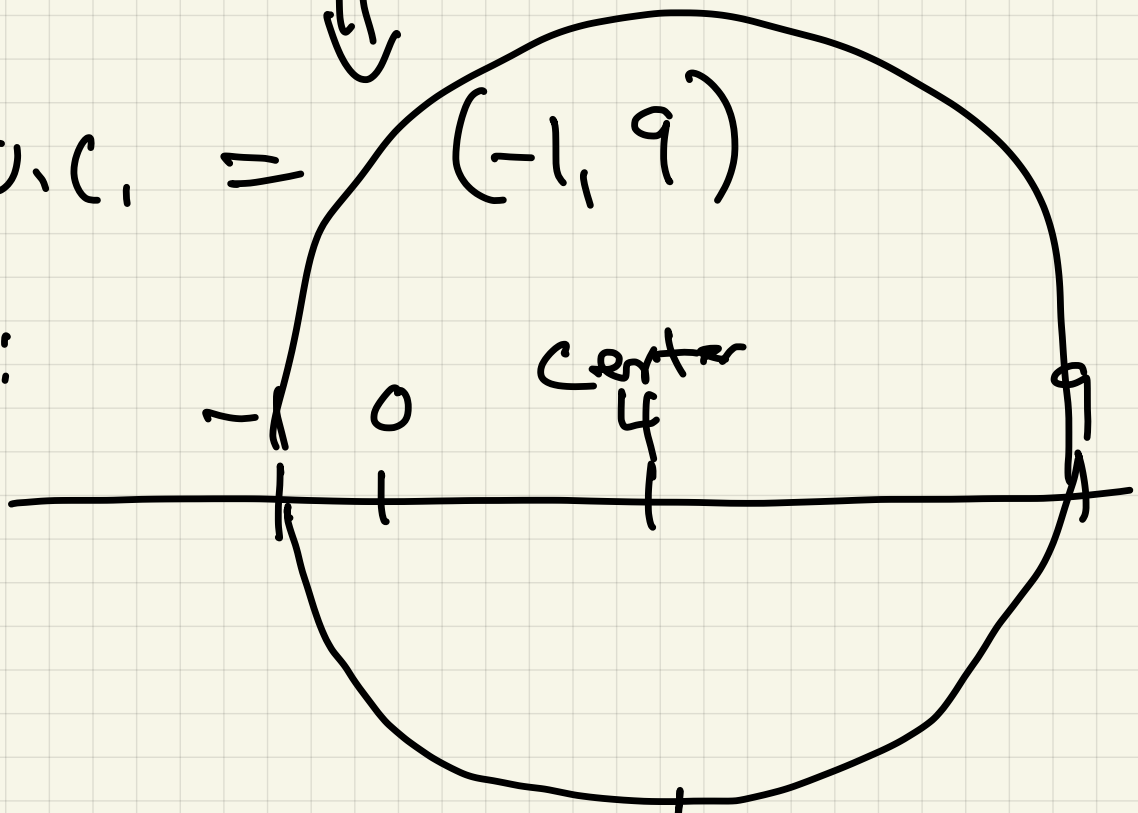
$x < -1,\ x > 9$ div

$x = -1$ $\sum (-1)^n$ div $\lim \neq 0$

$x = 9$ $\quad \sum 1^n$ $\quad$ div.

$\Downarrow$

I.O.C. $= (-1, 9)$



Rmk:

Also: The original power series

$$\left(\sum_{n=0}^{\infty} \frac{(x-4)^n}{5^n} \right)$$

$$1 + \frac{x-4}{5} + \frac{(x-4)^2}{5^2} + \cdots$$

is geometric series, so value "="

$$\frac{1}{1-\frac{x-4}{5}} = \frac{5}{5-(x-4)} = \frac{5}{9-x}$$

Ex 2 (a) $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = f(x)$

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| =$

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

all x

$\therefore$ I.O.C. $(-\infty, \infty)$

(b) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n \sqrt{n+10}}$

Ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{3^{n+1}\sqrt{n+11}}}{} \right| =$

$$\left| \frac{(x+1)^n}{3^n\sqrt{n+10}} \right|$$

$$\lim_{n\to\infty} \frac{|x+1|}{3} \sqrt{\frac{n+10}{n+11}} = \frac{|x+1|}{3} < 1$$

$$|x+1| < 3$$

$$\boxed{|x+1| < 3}$$

$$-3 < x+1 < 3$$

$$-4 < x < 2$$

End points:

$x = -4$: $\sum \frac{(-3)^n}{3^n\sqrt{n+10}} =$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+10}}}$$

$\frac{1}{\sqrt{n+10}} \to 0$

$\frac{1}{\sqrt{n+10}}$ dec $\Rightarrow$

conv. by A.S.T.

$x = 2$

$$\sum \frac{3^n}{3^n\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

divergent!

LCT with $\frac{1}{n}$ div. p-series

So I.O.C. $[-4, 2)$

General Picture:

A power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

converges in one of 3 ways:

(A) There's a number $R > 0$:

$f(x)$ converges for $|x-c| < R$

diverges for $|x-c| > R$

(B) $f(x)$ converges for all x $(R = \infty)$

(C) $f(x)$ converge for $x = c$ $(R = 0)$

R = radius of convergence