8/5/ Calc2

Partial fractions $\displaystyle\int \frac{N(x)}{D(x)}\,dx$

N, D polynomials, $\deg N < \deg D$

1. Factor $D(x)$ into linear

$\boxed{\begin{matrix} ax+b \\ \text{linear} \end{matrix}}$

ax^2+bx+c

$\boxed{\text{quadratic irreducible}}$

$x^2-4 = (x-2)(x+2)$

x^2+4 irreducible

2. for linear factor $(ax+b)^m$

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

3. for each quadratic $q = (ax^2+bx+c)^n$

include

$$\frac{B_1x+C_1}{q} + \frac{B_2x+C_2}{q^2} + \dots + \frac{B_nx+C_n}{q^n}$$

4) Solve constants A_i B_i C_i

5) Integrate

Ex 0 $\displaystyle\int \frac{6x^2+x-10}{x(x^2+5)}\,dx$

$$\frac{6x^2+x-10}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$$

Solved: $A=-2$, $B=8$, $C=1$

$$\int -\frac{2}{x} + \frac{8x+1}{x^2+5}\,dx$$

$$\frac{8x}{x^2+5} + \frac{1}{x^2+5}$$

$$= 4\left(\frac{2x}{x^2+5}\right)$$

$$-2\ln|x| + 4\ln|x^2+5| + \frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}} + C$$

Rmk: x^2+5 irreducible

x^2-4 $\quad (x-2)(x+2)$

In general:

$ax^2+bx+c=0 \Rightarrow$ quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

factors $\Leftrightarrow b^2-4ac \geq 0$

$\therefore$ irreducible $\Leftrightarrow b^2-4ac < 0$

$\underset{a}{1}x^2 + \underset{b}{3}x + \underset{c}{7}$

$b^2-4ac =$

$9-4(1)(7) =$

$-19 < 0$

Ex1 Write form of partial fractions

(a) $\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} =$

$$\frac{1}{(x-1)(x+1)(x^2+1)} \Rightarrow$$

$$\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Notice: # constants = degree of denominator

(b) $$\frac{5}{(x^2+2x-3)^2(x^2+2x+3)^2}$$

$x^2+2x-3 = (x+3)(x-1)$

$x^2+2x+3 =$ irreducible

$a=1,\ b=2,\ c=3$

$$b^2-4ac = 4-4(1)(3) = -8$$

$$\frac{5}{(x+3)^2(x-1)^2(x^2+2x+3)^2}$$

$$\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+$$

$$\frac{Ex+F}{x^2+2x+3}+\frac{Gx+H}{(x^2+2x+3)^2}$$

(c) $$\frac{1}{x^8-64x^2} = \frac{1}{x^2(x^6-64)} =$$

$$\frac{1}{x^2(x^3-8)(x^3+8)} =$$

$(x+2)(x^2-2x+4)$

$$\frac{1}{(x-2)(x^2+2x+4)}$$

$$\frac{1}{x^2(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)}$$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{Dx+E}{x^2+2x+4}+\frac{F}{x+2}+$$

$$\frac{Gx+H}{x^2-2x+4}$$

<u>Ex2</u> $\displaystyle\int \frac{8x^3+4}{x^4-1}$

$$\frac{8x^3+4}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$8x^3+4 = A(x+1)(x^2+1)+B(x-1)(x^2+1)$$

$$+\,C(x+D)(x-1)(x+1)$$

$$\| \qquad\qquad \|$$

$$\underbrace{(A+B+C)}_{8}x^3+\underbrace{(A-B+D)}_{0}x^2+\underbrace{(A+B-C)}_{0}x+\underbrace{(A-B-D)}_{4}$$

$$x = 1 \Rightarrow 12 = 4A \Rightarrow A = 3$$

$$x = -1 \Rightarrow -4 = -4B \Rightarrow B = 1$$

$$A = 3, B = 1, C = 4$$

$$A = 3, B = 1 \qquad 2 + D = 0$$

$$D = -2$$

So

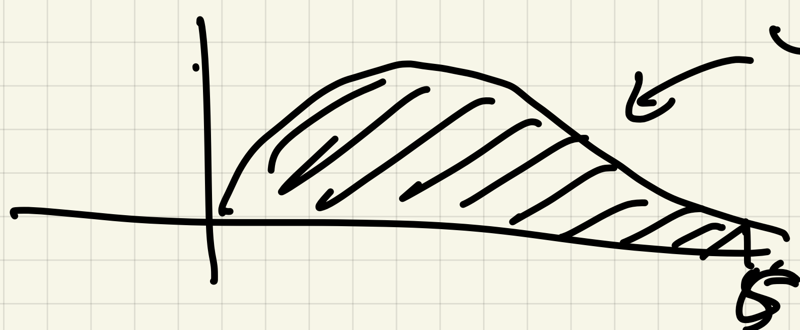
$$\int \frac{8x^3+4}{x^4-1} = \int \frac{3}{x-1} + \frac{1}{x+1} + \frac{4x-2}{x^2+1}$$

$$\int \frac{3}{x-1} + \int \frac{1}{x+1} + \int 2\frac{2x}{x^2+1} + \int \frac{-2}{x^2+1}$$

$$3\ln|x-1| + \ln|x+1| + 2\ln|x^2+1|$$

$$-2\arctan x + C$$

<u>Ex 3</u>



$$y = \frac{2x}{x^2+4}$$

Find area, volumes of revolution ~~of~~ about axes,

Area $= \int_0^5 \frac{2x}{x^2+1}\,dx =$

$\ln|x^2+1|\Big|_0^5 = \ln 26 + 1$

Volume about y-axis

$V = \int_0^5 2\pi x\left(\frac{2x}{x^2+1}\right)dx =$

$\int_0^5 2\pi \frac{2x^2}{x^2+1}\,dx =$

$2\pi \int_0^5 \frac{2x^2+2}{x^2+1} - \frac{2}{x^2+1}\,dx =$

$$2\pi\int_0^5 2 - \frac{2}{x^2+1}\,dx =$$

$$2\pi\left(2x - 2\arctan x\right)\Big|_0^5 =$$

$$2\pi\left(10 - 2\arctan 5\right) =$$

$$20\pi - 4\pi\arctan 5.$$

Volume about x-axis

Disks: $$V = \int_0^5 \pi\left(\frac{2x}{x^2+1}\right)^2 dx$$

$$= 4\pi\int_0^5 \frac{\cancel{4}x^2}{(x^2+1)^2}\,dx$$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{x^2}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1)+(Cx+D)=1x^2$$

$$\underbrace{A}_{0}x^3+\underbrace{B}_{1}x^2+\underbrace{(A+C)}_{0}x+\underbrace{(B+D)}_{0}=1x^2$$

$A=0$ $B=1$, $C=0$, $D=-1$

so $4\pi\int\frac{x^2}{(x^2+1)^2}dx=$

$$4\pi\int_0^5\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}dx$$

$$\int\frac{1}{(x^2+1)^2}dx \qquad \boxed{x=\tan\theta}$$

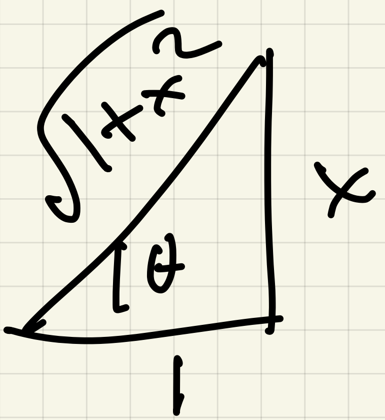
$$dx=\sec^2\theta$$

$$\int\frac{\sec^2\theta}{\sec^4\theta}dx \qquad 1+\tan^2=\sec^2$$

$$\int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} =$$

$$\frac{1}{2}\theta + \underline{\sin 2\theta}$$

$$\frac{1}{2}\theta + 2\sin\theta\cos\theta$$



$$\frac{1}{2}\arctan x + 2\,\frac{x \cdot 1}{1+x^2}$$

$$= \boxed{\frac{1}{2}\arctan x + \frac{2x}{1+x^2}}$$

$\int_0$

$$V = 4\pi\left(\arctan x - \frac{1}{2}\arctan x - \frac{2x}{1+x^2}\right)\Bigg|_0^5$$

$$4\pi\left(\frac{1}{2}\arctan 5 - \frac{10}{26}\right)$$

Improper integrals:

L'Hospital's Rule:

If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

If lim exists

Ex) (a) $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} \overset{LH}{=} \lim_{x \to 2} \frac{3x^2}{2x} = \frac{12}{4} = 3$

(b) $\lim_{x \to 0} \frac{x}{e^{5x} - 1} \overset{LH}{=} \lim_{x \to 0} \frac{1}{5e^{5x}} = \frac{1}{5}$

(c) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$ ✓

(d) $\lim_{x \to 0} \frac{\cos x - 1}{x^2} \overset{LH}{=}$

$\lim_{x \to 0} \frac{-\sin x}{2x} \overset{LH}{=} \lim_{x \to 0} \frac{-\cos x}{2} = -\frac{1}{2}$