3/26 | Calc2 — Quiz 13

$\boxed{1}$ $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$ $p=3$ conv to $\frac{1}{2}$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = -\frac{1}{2x^2}\Big|_1^b =$$

$$\lim_{b\to\infty} -\frac{1}{2b^2} - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$-\frac{1}{2b^2} \to 0$

$\boxed{2}$ $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}}$ $\quad x^{-1/2}$ $\quad p = \frac{1}{2} \leq 1$ $\Downarrow$ diverges

$$= \lim_{b\to\infty} 2x^{1/2}\Big|_1^b = +\infty$$

DNE

<u>diverges</u>

$\boxed{3}$ $\int_1^\infty \frac{1}{x+1} - \frac{2}{x+3} + \frac{1}{x+5}\,dx =$

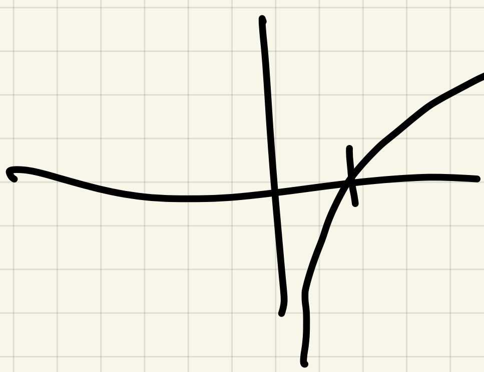
$$\lim_{b\to\infty} \int_1^b \frac{1}{x+1} - \frac{2}{x+3} + \frac{1}{x+5}\,dx$$

$$\ln|x+1| - 2\ln|x+3| + \ln|x+5| \Big|_1^b$$

$$\ln|(x+1)(x+5)| \qquad - \ln|x+3|^2$$

$$\lim_{b\to\infty} \ln\left|\frac{(x+1)(x+5)}{(x+3)^2}\right| \Bigg|_1^b =$$

$$\lim_{b\to\infty} \ln\left|\frac{(b+1)(b+5)}{(b+3)^2}\right| - \ln\left|\frac{2\cdot 6}{4^2}\right|$$



$$\lim_{b\to\infty} \frac{b^2+6b+5}{b^2+6b+9} = 1$$

$$= -\ln\left|\frac{12}{16}\right| = + \ln\left|\frac{16}{12}\right| = \ln\frac{4}{3}.$$

<u>Last time</u>: Infinite series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_n + \ldots$$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

$$\sum_{k=1}^{\infty} a_k = \underbrace{\lim_{n\to\infty} S_n}_{\S 9.1}$$

<u>Telescoping series</u>: sometimes

can calculate partial sums using fractions

<u>Geometric series</u> has form

$$a + ar + ar^2 + ar^3 + \ldots$$

$$\overset{\text{"}}{\sum_{k=0}^{\infty} ar^k}$$

first ~~term~~

r = common ratio

<u>Ex 0</u> $.7 + .007 + .00007 + \ldots$ $\overset{\text{"}}{\ }$ $\frac{70}{99}$

$a = .7 \quad r = .001$

<u>Ex 1</u> (a) $1 + 2/5 + 4/25 + 8/125 + 16/625 + \ldots$

$a = 1, \quad r = 2/5$

(b) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$

$a = 2, \; r = -1/2$

(c) $2 - 6 + 18 - 54 + 162 - \ldots$

$a = 2, \; r = -3$

Can compute partial sums:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \ldots$$

$$S_n = a + \boxed{ar + ar^2 + \ldots ar^{n-1}}$$

$$rS_n = \boxed{ar + ar^2 + ar^3 + \ldots + ar^{n-1}} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$\Downarrow$$

subtract?

$$(1-r)S_n$$

so $S_n = \dfrac{a(1-r^n)}{(1-r)}$ $\to 0$

Behavior

① $-1 < r < 1 \Rightarrow \lim_{n\to\infty} S_n = \frac{a}{1-r}$

② $r \geq 1$
$r \leq -1$ $\Rightarrow \lim r^n$ DNE

$\underline{r=1}$ $a+a+a+a+a+\ldots$
diverges

Conclusion:

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \begin{cases} \frac{a}{1-r} & |r|<1 \\ \text{diverges} & |r|\geq 1 \end{cases}$$

Example 2:

(a) $.7 + .007 + .00007 + \ldots$

$a = .7$
$r = .001$ $\Rightarrow$ converges to

$$\frac{.7}{1-.001} = \frac{.7}{.999} = \frac{700}{999}$$

(b) $1 + 2/5 + 4/25 + \ldots$ converges to

$$\frac{1}{1 - 2/5} = 5/3$$

(c) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots$

$r = -\frac{1}{2}$ converges to

$a = 2$

$$\frac{2}{1-(-\frac{1}{2})} = 2/(3/2) = \frac{4}{3}$$

(d) $2 - 6 + 18 - 54 \ldots$

$r = -3$ diverges.

(e) $\displaystyle\sum_{k=0}^{\infty} \frac{3}{4^k}$ $3 = a$, $r = \frac{1}{4}$ $\Rightarrow$

conv to $\dfrac{3}{1 - \frac{1}{4}} = \dfrac{3}{(3/4)} = 4$

(f) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{6^k} = \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \ldots$

$r = 1/6$ $a = 1/216$

conv to $\dfrac{1/216}{1-1/6} =$

$$\frac{\frac{1}{\cancel{216}_{36}}}{\frac{5}{\cancel{6}}} = \frac{1}{5\cdot 36} = \frac{1}{180}$$

$$(g) = \sum_{k=1}^{\infty} \frac{10^{2k}}{9^k \cdot 11^k} = \frac{100}{99} + \frac{10000}{9^2\cdot 11^2} + \ldots$$

$r = \frac{100}{99} > 1$ diverges

(h) $\sum_{k=0}^{\infty} \frac{1}{5^k} + \frac{1}{3^k}$

$$(1+1) + \left(\frac{1}{5}+\frac{1}{3}\right) + \left(\frac{1}{25}+\frac{1}{9}\right) + \ldots$$

NOT telescoping / geometric

BUT

$\sum_{k=0} \frac{1}{5^k}$ conv to $\frac{1}{1-1/5}\left(s_n \to \frac{5}{4}\right)$

$\sum_{k=0} \frac{1}{3^k}$ conv to $\frac{1}{1-1/3}$ ($t_n \to 3/2$)

so $s_n + t_n \to \frac{5}{4} + \frac{3}{2} = \frac{11}{4}$

partial sums for series

<u>Can combine series in general:</u>

If $\sum_{k=1}^{\infty} a_k = A$, $\sum_{k=1}^{\infty} b_k = B$, then

① $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

② $\sum a_k - b_k = A - B$

③ $\sum c a_k = cA$

No multiplication rule!!!

Ex 3 (a) $\sum_{k=1}^{\infty} \frac{3}{5^k} - \frac{8}{3^k}$

$a = 3/5$
$r = 1/5$

$a = 8/3$
$r = 1/3$

$\frac{3/5}{1-1/5} = \frac{3}{4}$

$\frac{8/3}{1-1/3} = 4$

conv to $3/4 - 4$

(b) $\sum_{k=1}^{\infty} \frac{18}{k^2+2k} - \frac{2^k}{9^k}$

Left side conv to $\frac{27}{2}$ (from Tuesday) right sides geometric:

$r = 2/9 \quad a = 2/9$

conv to $\frac{2/9}{1 - 2/9} = \frac{2}{7}$

so conv to $\frac{27}{2} - \frac{2}{7}$

Fact: If $\sum_{k=1}^{\infty} a_k$ converges,
then $\lim_{n\to\infty} a_n = 0$

Why? If $s_n = n^{\text{th}}$ partial sum

$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = L = \lim_{n\to\infty} s_{n-1}$

so

$$\lim_{n\to\infty} \underbrace{s_n - s_{n-1}}_{} = L - L = 0$$

$\lim_{n\to\infty} a_n$

Consequence:

nth term test (divergence test)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges

Ex 4 Apply nth term test:

(a) $\sum_{k=1}^{\infty} \frac{k}{1000k+50}$

$$\lim_{k\to\infty} \frac{k}{1000k+50} = \lim_{k\to\infty} \frac{1}{1000+\underbrace{\frac{50}{k}}_{\to 0}} = \frac{1}{1000} \neq 0$$

So $\sum a_k$ diverges

(b) $\sum_{k=2}^{\infty} \frac{1}{e^k}$ $\quad \lim_{k\to\infty} \frac{1}{e^k} = 0$

Test fails: BUT

$$\frac{1}{e^2}+\frac{1}{e^3}+\frac{1}{e^4}+\cdots$$

geometric series $a = \frac{1}{e^2}$

$r = \frac{1}{e} < 1$

$\therefore$ converges to

$$\frac{\frac{1}{e^2}}{1-\frac{1}{e}} = \frac{\frac{1}{e}}{e-1} =$$

$$\frac{1}{e(e-1)} = \frac{1}{e^2-e}$$

(c) $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

$$\lim_{k\to\infty} \ln\left(\frac{k+1}{k}\right) = 0$$

Test fails.

BUT

$$\ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \dots$$

$$\ln\left(\frac{n+1}{n}\right)$$

$$\cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \ln 4 - \cancel{\ln 3}$$

$$- \quad - \quad + \ln(n+1) - \cancel{\ln n}$$

$$S_n = -\ln 1 + \ln(n+1)$$

Diverges.