3/24/Calc 2

Last time

sequences $\{a_n\}$ $n \geq 1$

convergence

$\lim_{n\to\infty} a_n = L$ $\quad a_n \to L$

Can use calculus to compute limits when applicable

(A) Sandwich Theorem

a_n, b_n, c_n $\quad a_n \leq b_n \leq c_n$

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n \longrightarrow$

$\lim_{n\to\infty} b_n = L$

(B) $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$

(C) Combinations

Monotonic sequences

$a_1 \leq a_2 \leq a_3 \leq \ldots$

OR

$a_n \geq a_2 \geq a_3 \geq \dots$ there's M:

<u>Bounded</u> <u>above</u> $a_n \leq M$ all n

<u>below</u> There's L:

$L \leq a_n$

<u>Thm</u> $\{a_n\}$ is monotonic, then

$\{a_n\}$ converges $\Leftrightarrow$ $\{a_n\}$ is bounded

<u>Ex 1</u> $a_n = \frac{n}{n+1}$

(a) $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ ...

monotonic sequence

<u>Why?</u>, $a_n \leq a_{n+1}$

(1) $\frac{n}{n+1} \leq \frac{n+1}{n+2}$ for all n

$\Updownarrow$

$n(n+2) \leq \underline{(n+1)(n+1)}$

$n^2 + 2n \leq n^2 + 2n + 1$

② $f(x) = \frac{x}{x+1}$ $f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \frac{1}{(x+1)^2} > 0$

$\frac{n}{n+1} \le 1$ upper bound

$\therefore$ converges $\therefore$

$\lim_{x \to \infty} \frac{x}{x+1} \overset{LH}{=} \lim_{x \to \infty} \frac{1}{1} = 1$ ✓

(b) = $b_n = 2^n$

2 4 8 16 32 . . . —

monotonic ✓
unbounded,
not convergent.

(c) = $c_n = \frac{n^{60}}{e^n}$ $n \geq 0$

$c_0 = 0$, $c_1 = \frac{1}{e} \approx .368$, $c_2 \approx 1.56 \times 10^{17}$

$e_3 = 2.41 \times 10^{27}$ $e_4 = 2.43 \times 10^{34}$

Calc:

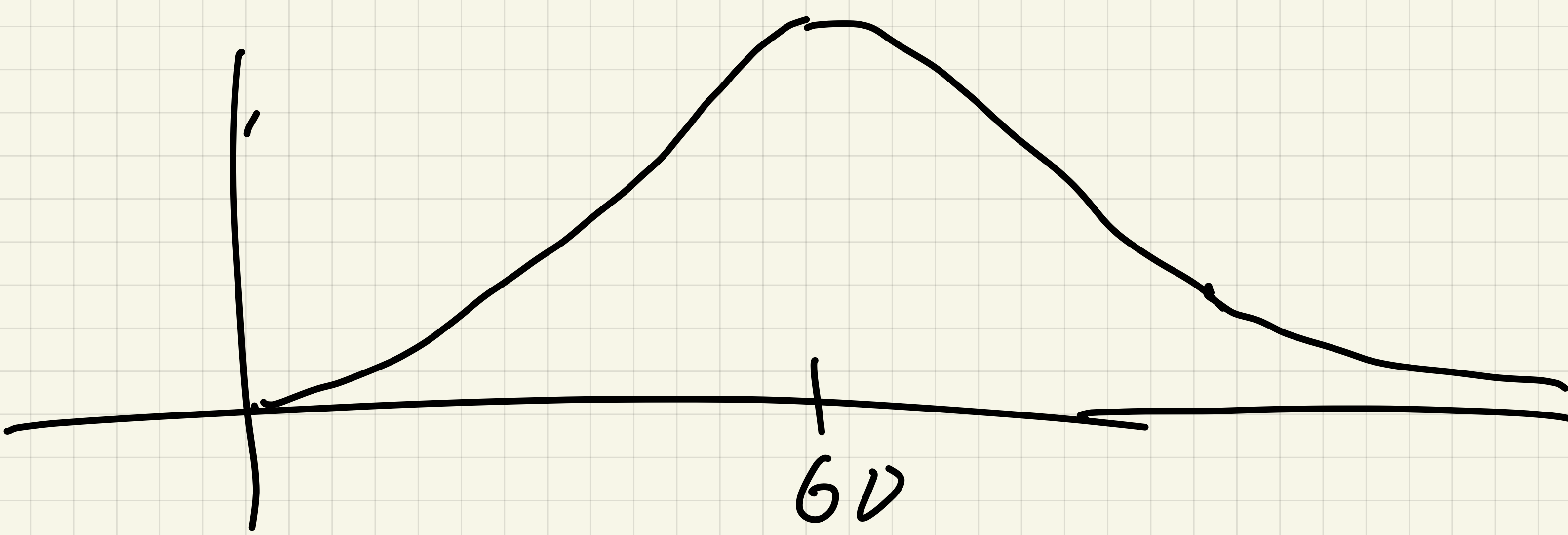
$$f(x) = \frac{x^{60}}{e^x}$$

$$f'(x) = \frac{e^x \cdot 60x^{59} - e^x \cdot x^{60}}{(e^x)^2}$$

$$= \frac{x^{59}(60-x)}{e^x}$$

$f'(x) > 0$ $x \leq 60$

$f'(x) < 0$ $x > 60$



NOT monotonic

Convergence

$$\lim_{x\to\infty} \frac{x^{60}}{e^x} \overset{LH}{=} \lim \frac{60x^{59}}{e^x} \overset{LH}{=}$$

$$\lim_{x\to\infty} \frac{60\cdot 59\cdot x^{58}}{e^x} = \cdots$$

$$\lim_{x\to\infty} \frac{60\cdot 59 \cdots 1}{e^x} = 0$$

~~lower~~ $\lim_{n\to\infty} \frac{n^{60}}{e^n} = 0$

lower bound 0

upper bound $\frac{60^{60}}{e^{60}} = 4.28\times 10^{80}$

no monotonic

9.2

Definition: If $\{a_n\}, n\geq 1$ is a sequence, then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$$

is an **infinite series**

with terms a_n

When does this make sense?

Set $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$

n^{th} partial sum

If $\lim_{n \to \infty} s_n = L$, then

$\sum_{k=1}^{\infty} a_k$ **converges to L**

If $\lim_{n \to \infty} s_n$ DNE, then

$\sum_{k=1}^{\infty} a_k$ **diverges**

Punchli $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$

Ex1 $a_k = 1$

$$1+1+1+\dots = \lim_{n\to\infty} s_n$$

$s_1 = 1$

$$= \lim_{n\to\infty} n = +\infty$$

$s_1 = 1, \; s_2 = 2, \; s_3 = 3$

Ex2 $a_k = \dfrac{5}{k(k+1)} \quad k \geq 1$

$$\frac{5}{2} + \frac{5}{6} + \frac{5}{12} + \frac{5}{20} + \frac{5}{30} + \dots$$

1 2

Look at partial sums

$$s_1 = \frac{5}{2}$$

$$S_2 = \frac{5}{2} + \frac{5}{6} = 5\left(\underbrace{\frac{1}{2} + \frac{1}{6}}\right) =$$

$$5\left(\frac{4}{6}\right) = 5\left(\frac{2}{3}\right)$$

$$S_3 = \underbrace{\frac{5}{2} + \frac{5}{6}} + \frac{5}{12}$$

$$5\left(\frac{2}{3}\right) + 5\left(\frac{1}{12}\right) =$$

$$5\left(\frac{2}{3} + \frac{1}{12}\right) = 5\left(\frac{8+1}{12}\right)$$

$$5\left(\frac{9}{12}\right) = 5\left(3/4\right)$$

Looks like $S_4 = 5\left(4/5\right) = 4$

Looks like: $S_n = 5\left(\frac{n}{n+1}\right)$

Let's confirm this guess:

$$a_k = \frac{5}{k(k+1)} = \frac{5}{k} - \frac{5}{k+1} \leftarrow$$

Partial fractions:

$$S_n = a_1 + a_2 + a_3 \ldots a_n$$

$$= \left(\frac{5}{1} - \cancel{\frac{5}{2}}\right) + \left(\cancel{\frac{5}{2}} - \cancel{\frac{5}{3}}\right) + \left(\cancel{\frac{5}{3}} - \frac{5}{4}\right) + \ldots + \left(\cancel{\frac{5}{n}} - \frac{5}{n+1}\right)$$

$$= \frac{5}{1} - \frac{5}{n+1} = \underline{\underline{5\left(\frac{n}{n+1}\right)}}$$

So $\sum a_k = \lim_{n \to \infty} S_n =$

$$\lim_{n \to \infty} 5\left(\frac{n}{n+1}\right) = 5$$

<u>Ex2</u> Evaluate $\sum_{k=1}^{\infty} \frac{18}{k^2+2k} =$

$$6 + \frac{9}{4} + \frac{6}{5} + \ldots$$

use same idea:

$$\frac{18}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} = \frac{9}{x} - \frac{9}{x+2}$$

Using this, partial sum:

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$= \left(\frac{9}{1} - \frac{9}{3}\right) + \left(\frac{9}{2} - \frac{9}{4}\right) + \left(\frac{9}{3} - \frac{9}{5}\right) + \cdots$$

$$\left(\frac{9}{4} - \frac{9}{6}\right) + \left(\frac{9}{5} - \frac{9}{7}\right) + \cdots \left(\frac{9}{n} - \frac{9}{n+2}\right)$$

$$\parallel$$

$$S_n = \frac{9}{1} + \frac{9}{2} - \frac{9}{n+1} - \frac{9}{n+2}$$

$$\sum a_k = \lim_{n \to \infty} S_n = \frac{9}{1} + \frac{9}{2} = \frac{27}{2}$$

Ex 3 $\sum_{k=0}^{\infty} \frac{7}{10^{2k+1}} = \frac{7}{10} + \frac{7}{1000} + \frac{7}{100000} + \cdots$

$$= .7 + .007 + .00007 + \cdots$$

$$= .707070707\ldots$$

$$= .\overline{70}$$

$$s = .707070\ldots$$

$$100s = 70.707070\ldots$$

$$100s - s = 70$$

$$\underbrace{100s - s}_{= 99s} \Rightarrow s = \frac{70}{99}$$

More generally, a geometric series has form

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

$$\left(a = \frac{7}{10},\ r = \frac{1}{100} \quad \text{gives sequence above} \right)$$

$$\sum_{k=0}^{\infty} ar^k$$