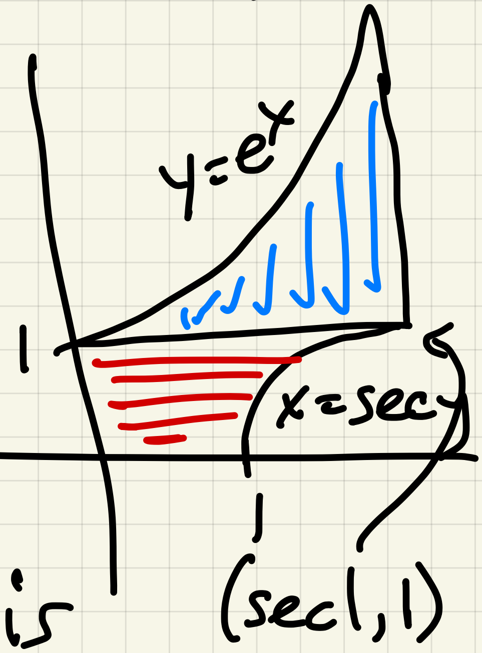
2/3/ Calc 2

Shells vs. Washers

Last time

Ex 1 R is region



Find volume of revolution about

(a) x-axis (b) y-axis

Note should handle top / bottom differently:

Washers Shells

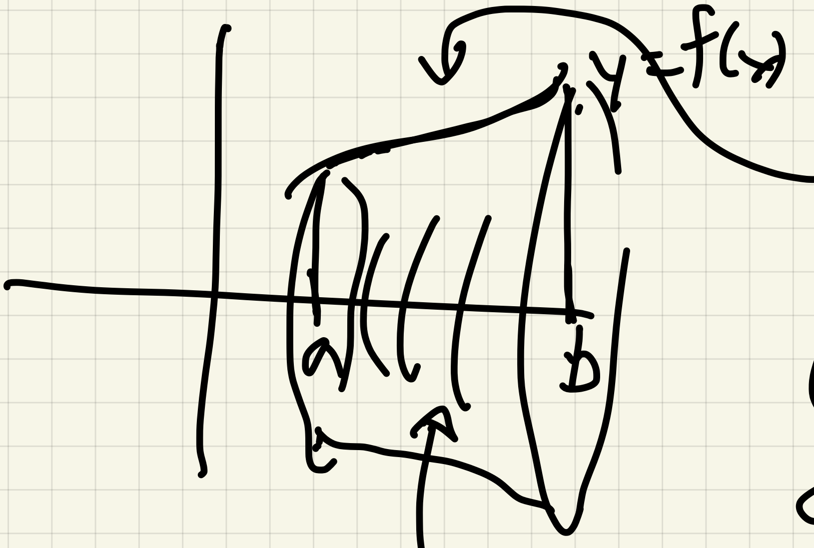
(a) $V = \int_0^{\sec 1} \pi\left((e^x)^2 - 1^2\right)dx + \int_0^1 2\pi y(\sec y)\,dy$

Shells Washers

(b) $V = \int_0^{\sec 1} 2\pi x(e^x - 1)\,dx + \int_0^1 \pi(\sec y)^2\,dy$

§6.3-6.4 Arc length and Surface area

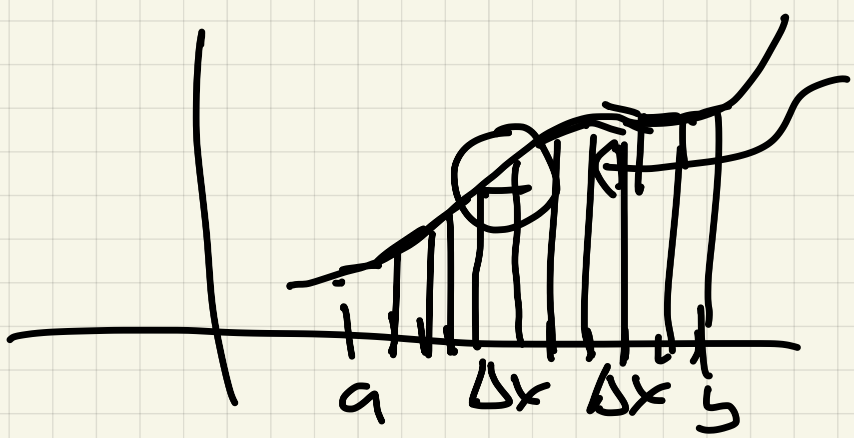
Two geometric questions:

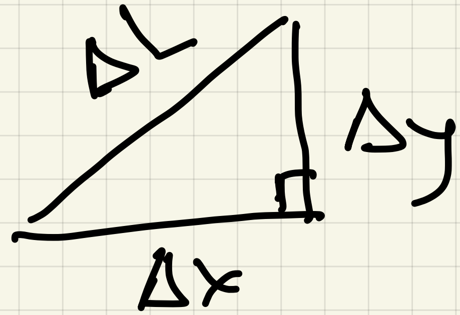


(A) What is length of curve?,

(B) What is surface area of revolution?,

To <u>estimate</u> both (A) & (B), Partion $[a,b]$ into n equal segments, connect dots, add lengths:





$$\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x \quad ,$$

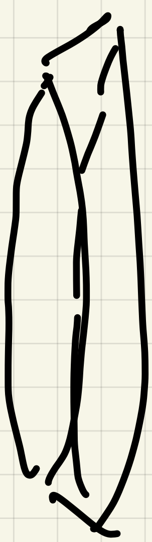
But $\frac{\Delta y}{\Delta x} \approx f(x_i)$ from Calc 1, so $\approx \sqrt{1+(f'(x_i))^2}\,\Delta x$

Therefore

(A) $L = \text{length} \approx \sum_{i=1}^{n} \Delta L \approx \sum_{i=1}^{n} \sqrt{1+f'(x_i)^2}\,\Delta x$

(B) $SA = \text{surface area} = \sum_{i=1}^{n} \left(\text{Area of circular strips}\right)$

$$= \sum_{i=1}^{n} \underbrace{2\pi f(x_i)}_{\text{circumf}} \underbrace{\sqrt{1+(f'(x_i))^2}\,\Delta x}_{\text{width}}$$



← ith strip, flatten it

To get <u>exact</u> answers,
take limit as $n \to \infty$ to get

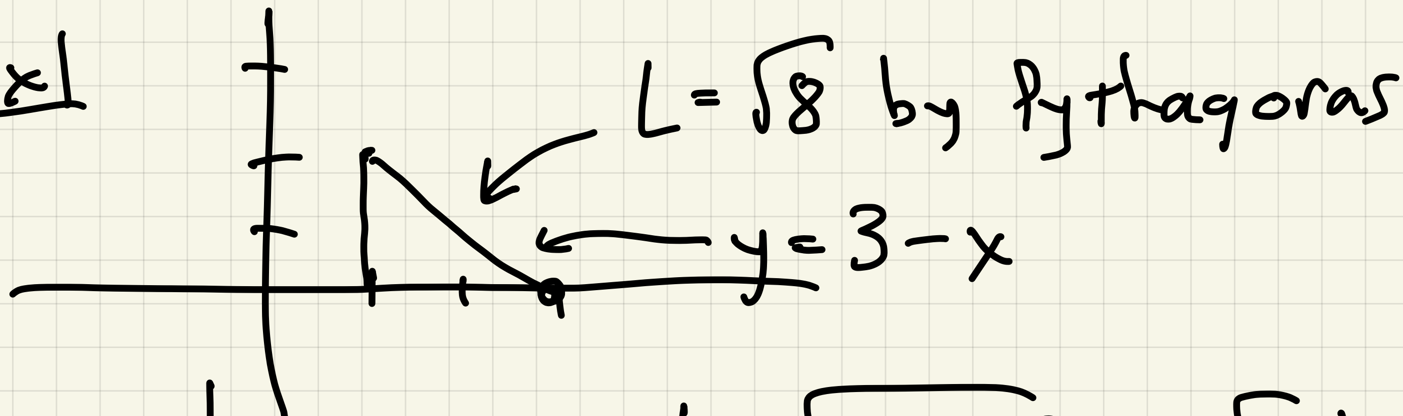
(A) Arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

(B) SA $= \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Notation: $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \Rightarrow$

$L = \int ds \qquad SA = \int 2\pi \underbrace{f(x)}_{\text{radius of strip}}\, ds$

Ex1



$\frac{dy}{dx} = -1 \Rightarrow ds = \sqrt{1+(-1)^2}\,dx = \sqrt{2}\,dx$

so $L = \int_1^3 \sqrt{2}\,dx = \sqrt{2}x\Big|_1^3 = 2\sqrt{2} = \sqrt{8}$ ✓

$SA = \int_1^3 2\pi(3-x)\underbrace{\sqrt{2}\,dx}_{ds} = 2\pi\sqrt{2}\int_1^3 (3-x)\,dx$

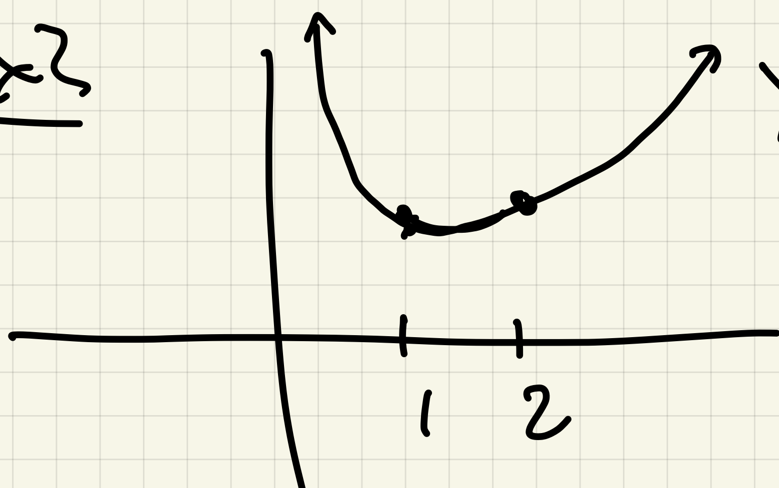
$= 2\pi\sqrt{2}\left(3x - \frac{x^2}{2}\right)\Big|_1^3 = 4\sqrt{2}\pi$

Easy because $ds = \sqrt{2}\,dx$.

Usually integrals trickier!

Often book problems feature following technique:

Ex 3



$$y = \frac{x^3}{6} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2},$$

so $$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} =$$

$$\sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} = \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} =$$

$$\sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2}$$ (square root is gone!)

So $$L = \int_1^2 ds = \int_1^2 \frac{x^2}{2} + \frac{1}{2x^2}\,dx =$$

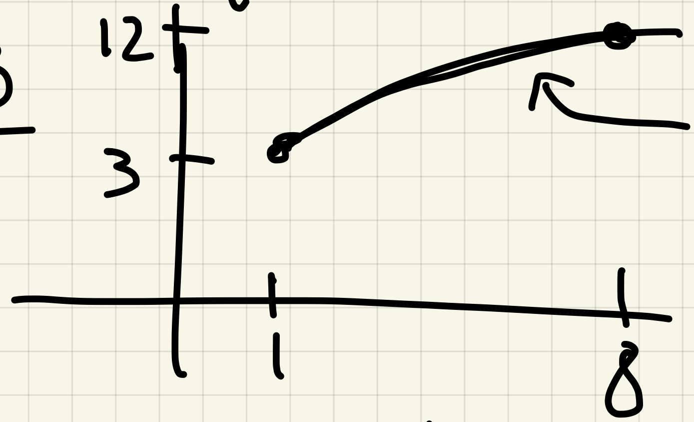
$$\frac{x^3}{6} - \frac{1}{2x}\Big|_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}$$

<u>SA about x-axis</u> $\int_1^2 2\pi\left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$

<u>SA about y-axis</u> $\int_1^2 2\pi \underbrace{x}_{\text{radius}} \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$

Integral can be tricky:

<u>Ex 3</u>



$y = 3x^{2/3}, \ 1 \le x \le 8$

<u>Arc length</u>: $\frac{dy}{dx} = 2x^{-1/3} \Rightarrow$

$$ds = \sqrt{1 + 4x^{-2/3}}\,dx = \sqrt{\frac{x^{2/3}+4}{x^{2/3}}}\,dx =$$

$$\frac{1}{x^{1/3}}\sqrt{x^{2/3}+4} = x^{-1/3}\sqrt{x^{2/3}+4}\,dx$$

so $L = \int_1^8 x^{-1/3}\sqrt{x^{2/3}+4}$ $\qquad u = x^{2/3}+4$

$\qquad\qquad\qquad\qquad\qquad\qquad du = \frac{2}{3}x^{-1/3}dx$

$\quad\ \ \,\|$

$$\frac{3}{2}\,du = x^{-1/3}\,dx$$

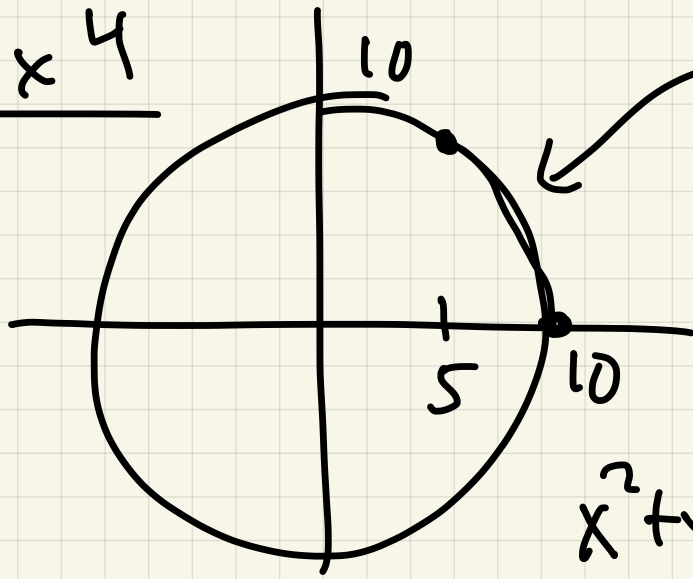
$$\int_{u=5}^{u=8} \frac{3}{2}\sqrt{u}\,du = u^{3/2}\Big|_5^8 = 8^{3/2} - 5^{3/2}$$

$$SA = \int_1^8 2\pi \cdot 3x^{2/3}\sqrt{1+4x^{-2/3}}\,dx =$$

$$6\pi\int_1^8 x^{1/3}\sqrt{x^{2/3}+4}\,dx = ??$$

NO IDEA!

Ex 4



(a) Arc length?

(b) SA of revolution about x-axis

$$x^2+y^2=100 \Rightarrow y=\sqrt{100-x^2}$$

so $\frac{dy}{dx} = \frac{-x}{\sqrt{100-x^2}}$, $ds = \sqrt{1+\left(\frac{-x}{\sqrt{100-x^2}}\right)^2}$

$$= \sqrt{1+\frac{x^2}{100-x^2}} = \sqrt{\frac{100-\cancel{x^2}+\cancel{x^2}}{100-x^2}} =$$

$$\sqrt{\frac{100}{100-x^2}}\,dx = \frac{10}{\sqrt{100-x^2}}\,dx$$

so $$L = \int_5^{10} ds = \int_5^{10} \frac{10}{\sqrt{100-x^2}}\,dx =$$

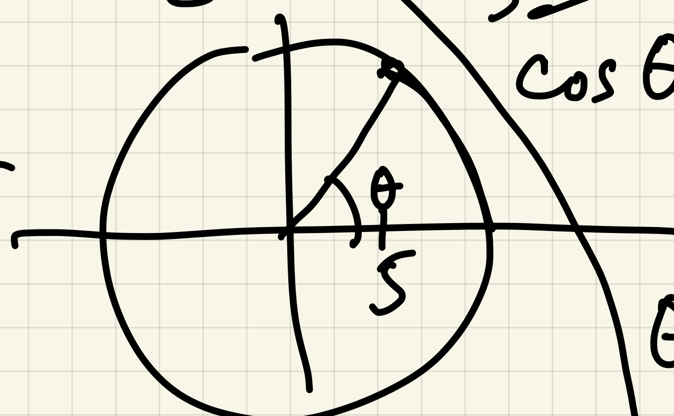
$$\left(\underline{\text{Recall}} \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \right)$$

$$10 \arcsin\frac{x}{10} \Big|_5^{10} = 10\,\underbrace{\arcsin 1}_{\pi/2} - 10\,\underbrace{\arcsin\tfrac{1}{2}}_{\pi/6}$$

$$= 10\left(\tfrac{\pi}{2} - \tfrac{\pi}{6}\right) = \frac{10\pi}{3}$$

same!

<u>Confirmation</u>



$\cos\theta = \frac{5}{10} = \frac{1}{2}$

$\Downarrow$

$\theta = \pi/3 = 60°$

$60° = 1/6$ of circle,

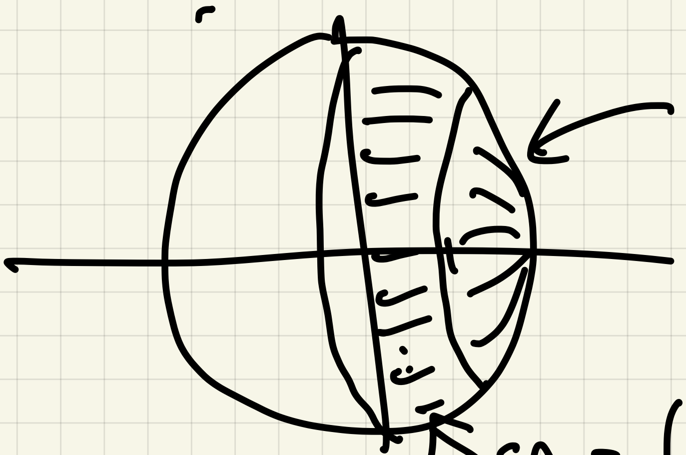
so $L = 1/6\,(\text{Circumference})$

$$= \frac{1}{6}(2\pi \cdot 10) = \frac{20\pi}{6} = \frac{10\pi}{3} \checkmark$$

(b) $$SA = \int_5^{10} 2\pi \cancel{\sqrt{100-x^2}} \cdot \underbrace{\frac{10}{\cancel{\sqrt{100-x^2}}}\,dx}_{ds} =$$

$$\int_5^{10} 2\pi \cdot 10\,dx = \int_5^{10} 20\pi\,dx = 20\pi x\Big|_5^{10} = 100\pi$$

<u>Notice</u>:



$SA = 100\pi$

same!

$SA = \int_0^5 20\pi\,dx = 100\pi$

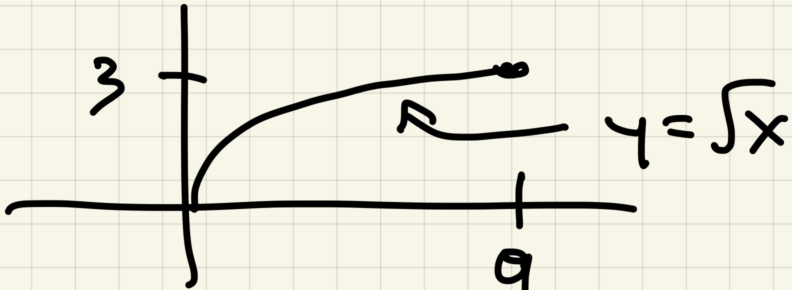
<u>Cool fact</u>: The SA over any curve of width 5 gives same surface Area 100π

Split $[-10,10]$ into $[-10,-5][-5,0]$ $[0,5][5,10]$ $\Rightarrow$

$$\text{Total SA} = 100\pi + 100\pi + 100\pi + 100\pi = 400\pi$$

(consistent with $SA = 4\pi r^2$ formula)

Ex5



(a) Arc length: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow ds = \sqrt{1 + \frac{1}{4x}}$,

so $L = \int_0^9 \sqrt{1 + \frac{1}{4x}}\, dx$ Hard integral!

(b) Can try alternate approach with y-variable: $x = y^2$,

$\frac{dx}{dy} = 2y \Rightarrow ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy =$

$\sqrt{1 + 4y^2}\, dy$, so

$L = \int_0^3 \sqrt{1 + 4y^2}\, dy$ Better, but still hard.

On the other hand, SA is easy!

(c) $SA = \int_0^9 2\pi \overbrace{\sqrt{x}\sqrt{1 + \frac{1}{4x}}}^{\text{combine}}\, dx =$

$$\int_0^9 2\pi\sqrt{x+\frac{1}{4}}\,dx = \qquad u = x + 1/4$$

$$du = dx$$

$$\int_{\frac{1}{4}}^{9\frac{1}{4}} 2\pi\sqrt{u}\,du = 2\pi\cdot\frac{2}{3}u^{3/2}\Big|_{1/4}^{37/4} =$$

$$\frac{4\pi}{3}\left(\left(\frac{37}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right) = \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$