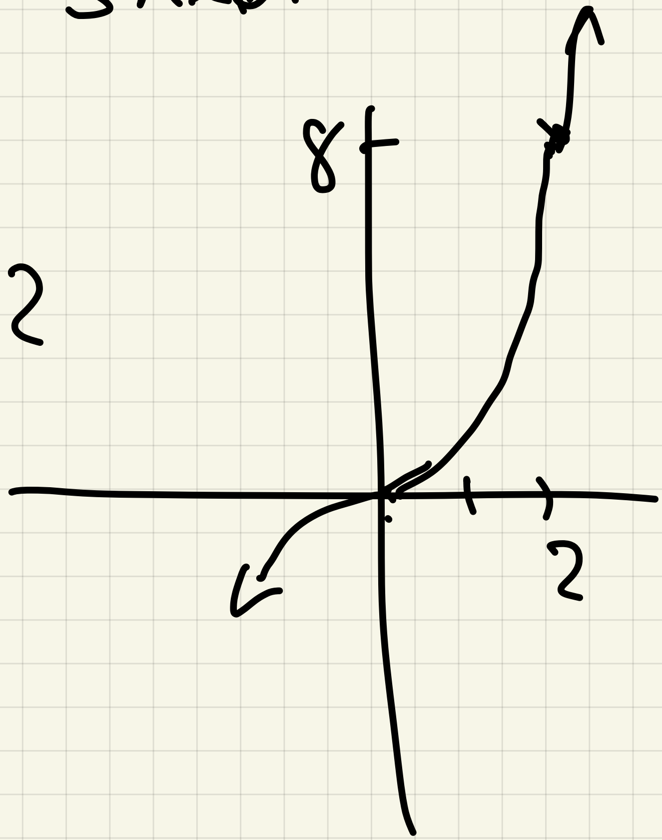
2/12/Calc2

Exam 1 → Thursday

Review sheet

Qn #5

$y = x^3 \quad 0 \le x \le 2$



$\boxed{1}$ $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\frac{dy}{dx} = 3x^2$

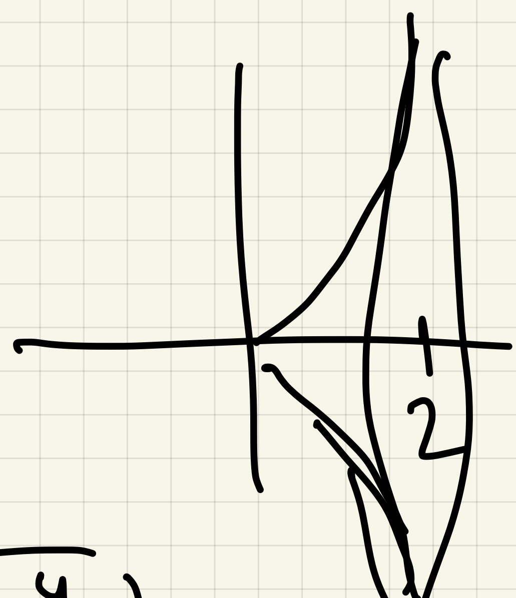
$$\int_0^2 \sqrt{1 + 9x^4}\, dx$$

$\boxed{2}$

$$SA = \int_0^2 2\pi (\text{rad})\, ds$$

y

||



$$\int_0^2 2\pi \underline{\underline{x}}^3 \sqrt{\underbrace{1+9x^4}}\, dx$$

$$u = 1 + 9x^4$$
$$du = 36\,x^3\,dx$$
$$\frac{1}{36}\,du = x^3\,dx$$

$$\underline{2\pi} \int_1^{145} \frac{1}{36} \sqrt{u}\,du = \frac{\pi}{\cancel{18}_9}\,\frac{\cancel{2}}{3}\,u^{3/2}\Big|_1^{145}$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

<u>BTW</u> You can use y variable:

$$x = y^{1/3} \qquad \frac{dx}{dy} = \frac{1}{3}\,y^{-2/3}$$

$$\int_0^8 2\pi y \sqrt{1 + \frac{1}{9}\,y^{-4/3}}\,dy =$$

$$2\pi \int_0^8 y\,\frac{1}{3}\sqrt{9 + \underset{\shortparallel}{y^{-4/3}}}\,dy$$

$$\left(y^{-2/3}\right)^2$$

$$\frac{2\pi}{3}\int_0^8 \boxed{y^{1}\, y^{-2/3}}\sqrt{9y^{4/3}+1}\,dy$$

$$\frac{2\pi}{3}\int_0^8 y^{1/3}\sqrt{9y^{4/3}+1}\,dy$$

$u = 9y^{4/3}+1$

$du = 12y^{1/3}\,dy$

$\frac{du}{12} = y^{1/3}\,dy$

$$\frac{2\pi}{3}\int_{u=1}^{145}\frac{1}{12}\sqrt{u}\,du = \frac{\pi}{27}\left(145^{3/2}-1\right)$$

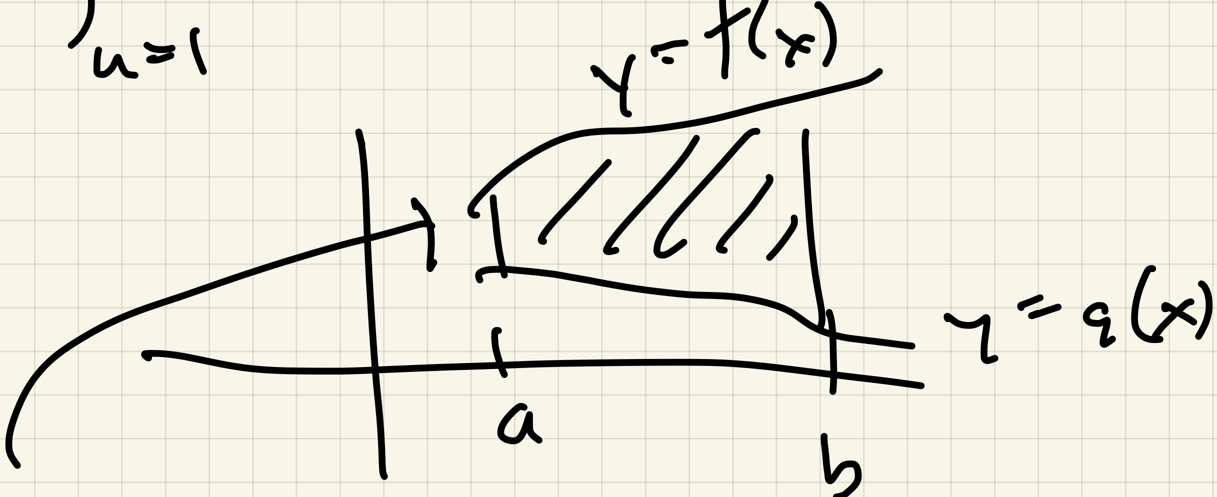


Plate has constant density δ,

Then

$$M = \int_a^b \delta\,(f(x) - g(x))\,dx$$

$$= \delta \int_a^b \underline{(f(x) - g(x))}\,dx$$

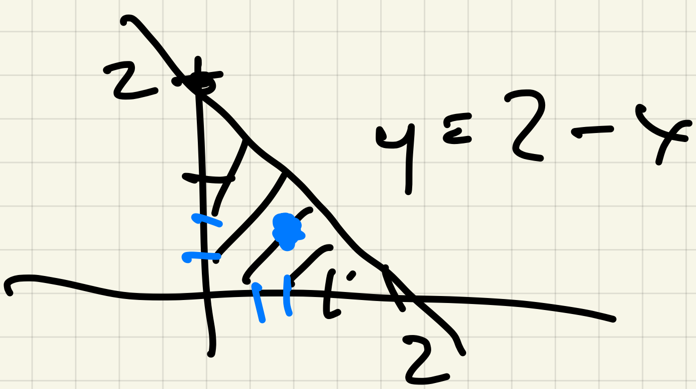
$$\left.\begin{aligned} M_y &= \delta \int_a^b (f(x) - g(x))\,x\,dx \\ M_x &= \delta \int_a^b \frac{f(x)^2 - g(x)^2}{2}\,dx \end{aligned}\right\}$$

$$COM = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M_y}, \frac{M_x}{M}\right)$$

<u>Rmk</u>: δ cancels in formula for ~~ce~~ COM:

($\mathrm{so}\ \delta = 1$) (Centroid)

<u>Ex 1</u>



Find centroid (COM):

$$M = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^2 =$$

$$(4-2) - 0 = 2 \checkmark$$

$$M_y = \int_0^2 (2-x)\,x\,dx =$$

$$\int_0^2 2x - x^2\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 =$$

$$4 - 8/3 = 4/3$$

$$M_x = \int_0^2 \frac{(2-x)^2 - 0^2}{2}\,dx$$

$u = 2-x$

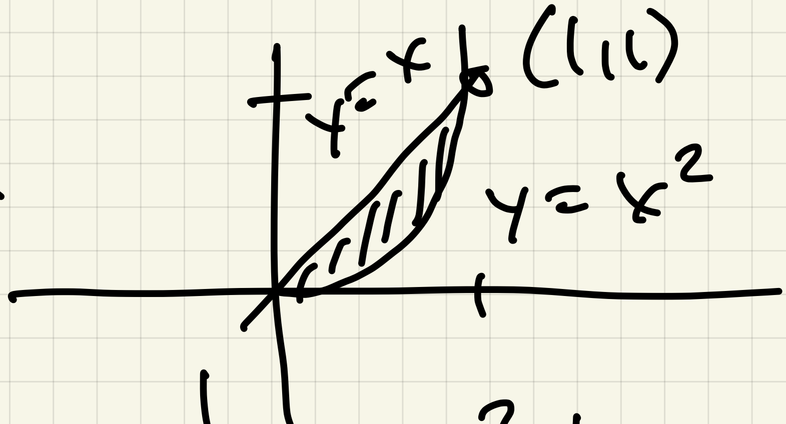
$du = -dx$

$$= -\int_2^0 \frac{u^2}{2}\,du =$$

$$-\frac{u^3}{6}\Big|_2^0 = 0 - \left(-8/6\right) = 4/3$$

so $COM = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{2}{3}, \frac{2}{3}\right)$

Ex 2



$$M = \int_0^1 x - x^2 \, dx = \frac{1}{2}x^2 - \frac{x^3}{3}\Big|_0^1 =$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$M_y = \int_0^1 x(x - x^2)\, dx = \int_0^1 x^2 - x^3 \, dx =$$

$$\frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{12}$$

$$M_x = \int_0^1 \frac{x^2 - (x^2)^2}{2} \, dx = \frac{1}{2}\int_0^1 x^2 - x^4 =$$

$$\frac{1}{2}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$$

$$\text{COM} = (\bar{x}, \bar{y}) = \left(\frac{1/12}{1/6}, \frac{1/15}{1/6} \right) =$$

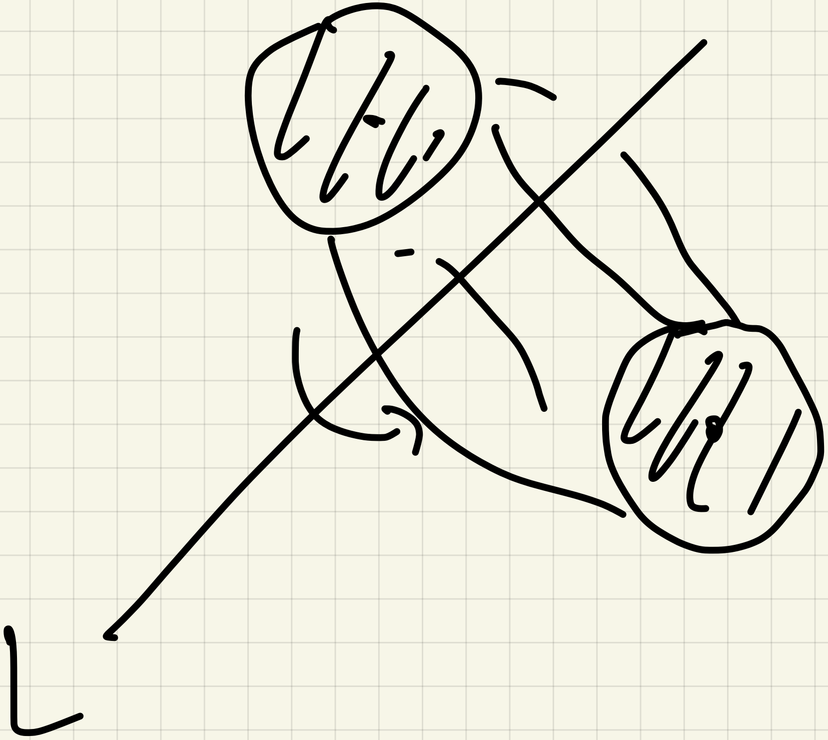
$$\left(6/12, 6/15 \right) = \left(\frac{1}{2}, 2/5 \right)$$

<u>Pappus (290-360 AD)</u>:

If R is a region with area A, and centroid $(\bar{x}, \bar{y})$, L is a line of distance r to $(\bar{x}, \bar{y})$.

Then volume of revolution of R about line L is

$$V = 2\pi r A$$



<u>Ex 3</u> For region R find volume of revolution about

a) y-axis

b) x-axis

c) $x=4$

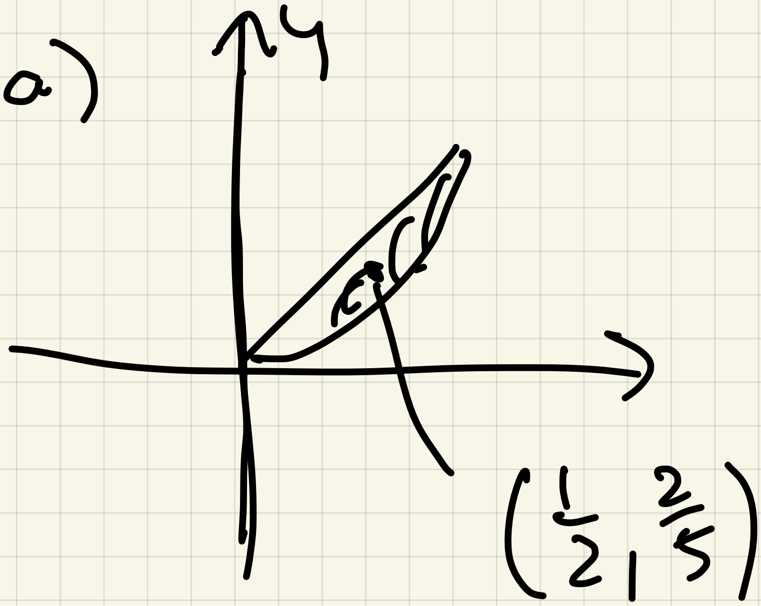
d) $y=2$

e) $x=-2$

f) $y=-1$

<u>Note</u>: with disks/shells 6 integrals

a)



$A = 1/6$

$\left(\frac{1}{2}, \frac{2}{5}\right)$ COM

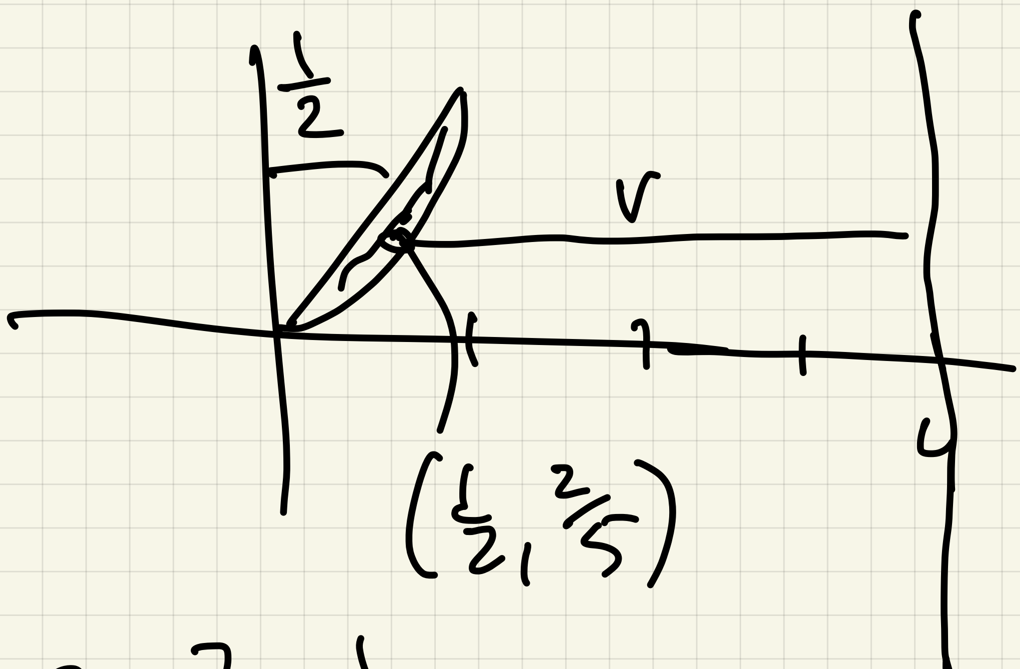
$r = \frac{1}{2}$

$$V = 2\pi \, \frac{1}{2} \cdot \frac{1}{6} = \frac{\pi}{6}$$

b) x-axis

$$V = 2\pi \, \frac{2}{5} \cdot \frac{1}{6} = \frac{4\pi}{30} = \frac{2\pi}{15}$$

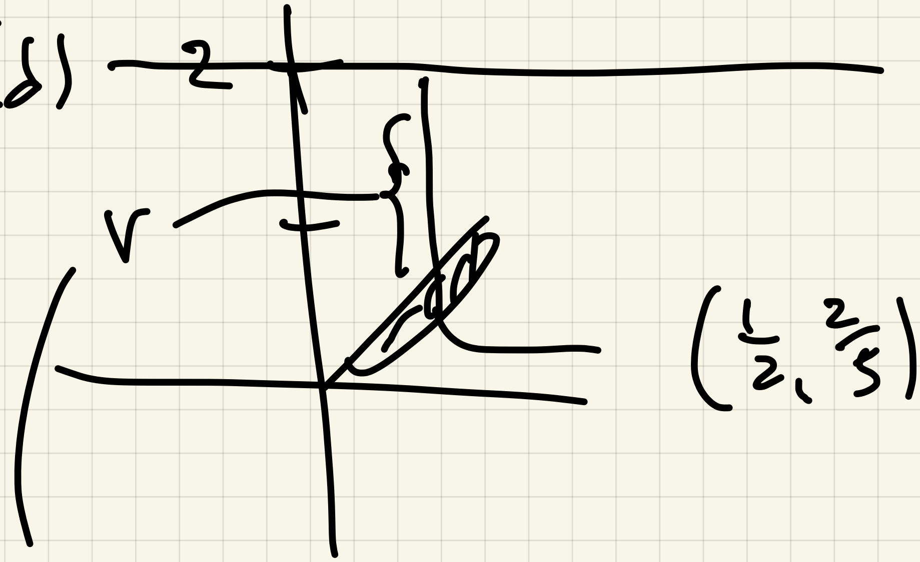
c) $\underline{x = 4}$



$$r = 4 - \frac{1}{2} = \frac{7}{2}$$

$$V = 2\pi \, \frac{7}{2} \cdot \frac{1}{6} = 7\pi/6$$

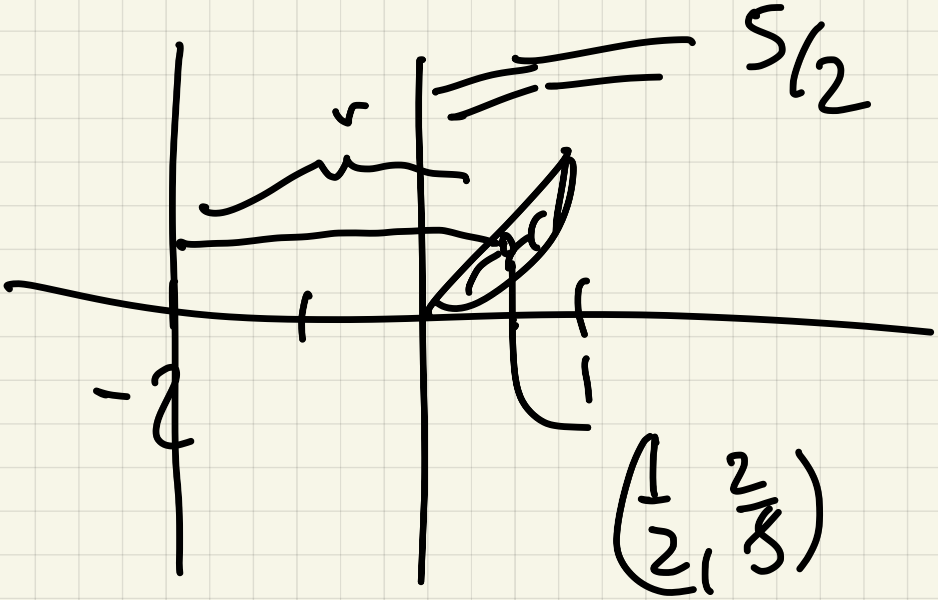
(d)



$2 - \frac{2}{5} = 8/5$, so

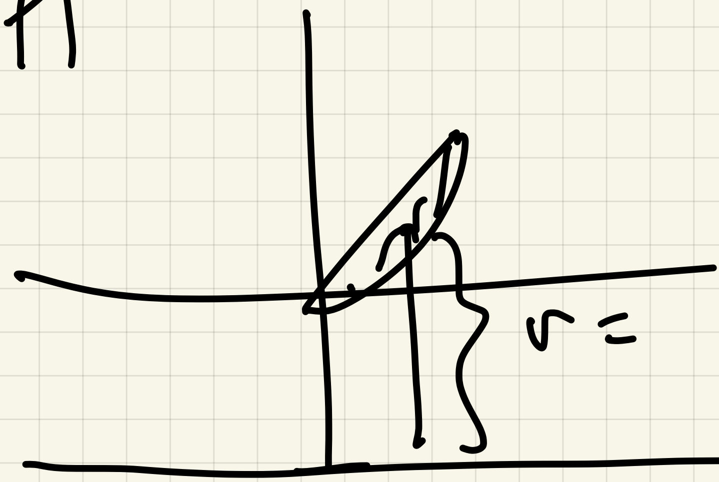
$$V = 2\pi \frac{8}{5} \cdot \frac{1}{6} = \frac{16\pi}{30} = \frac{8\pi}{15}$$

e) $x = -2$



so $V = 2\pi \frac{5}{2} \frac{1}{6} = \frac{10\pi}{12} = \frac{5\pi}{6}$

(f)

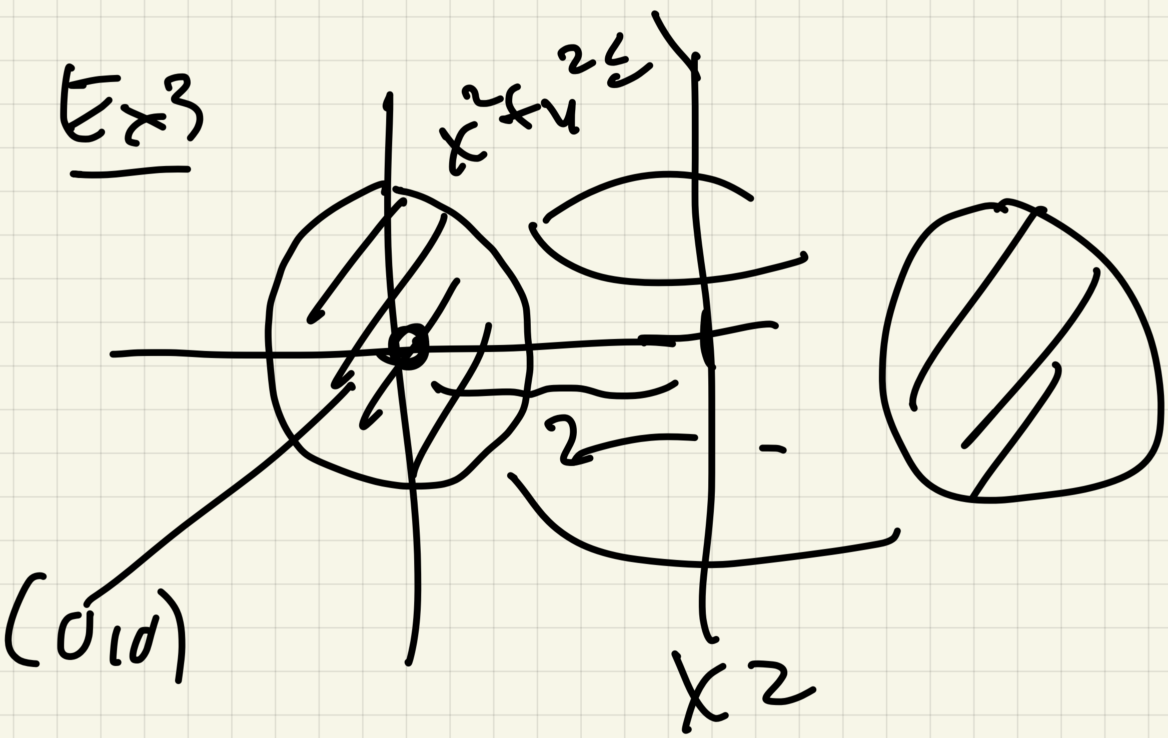


$r = 1 + 2/5 = 7/5$

$y = -1$

So $V = 2\pi \frac{7}{5} \frac{1}{6} = \frac{7\pi}{15}$

Ex3



$V = 2\pi r \pi = 2\pi \cdot 2\pi = 4\pi^2$

$r = 2$

Ex 4

(a) Find centroid;

$x^2+y^2=r^2$

(b) Find volume of revolution about y-axis

$$y=\sqrt{r^2-x^2}$$

$$M = \int_{-r}^{r}\sqrt{r^2-x^2}\,dx = \left(\frac{\text{Area of radius } r \text{ circle}}{2}\right)$$

$$= \frac{\pi r^2}{2} = M$$

$$M_y = \int_{-r}^{r} x\sqrt{r^2-x^2}\,dx$$

$u = r^2-x^2$

$du = -2x\,dx$

$$\int_0^0 -\frac{1}{2}\sqrt{u}\,du = 0$$

$$M_x = \int_{-r}^{r} \frac{\left(\sqrt{r^2-x^2}\right)^2 - 0^2}{2}\,dx =$$

$$\frac{1}{2}\int_{-2}^{r} r^2 - x^2\,dx = \frac{1}{2}\left(\underline{r^2x - \frac{x^3}{3}}\right)\Big|_{-r}^{r}$$

$$\frac{1}{2}\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right) =$$

$$\frac{1}{2}\left(\frac{4}{3}r^3\right) = \frac{2}{3}r^3 = M_x$$

so $\text{Com} = (\bar{x}, \bar{y}) = \left(\dfrac{0}{\frac{\pi r^2}{2}}, \dfrac{\frac{2}{3}r^3}{\frac{\pi r^2}{2}}\right)$

$= 0$

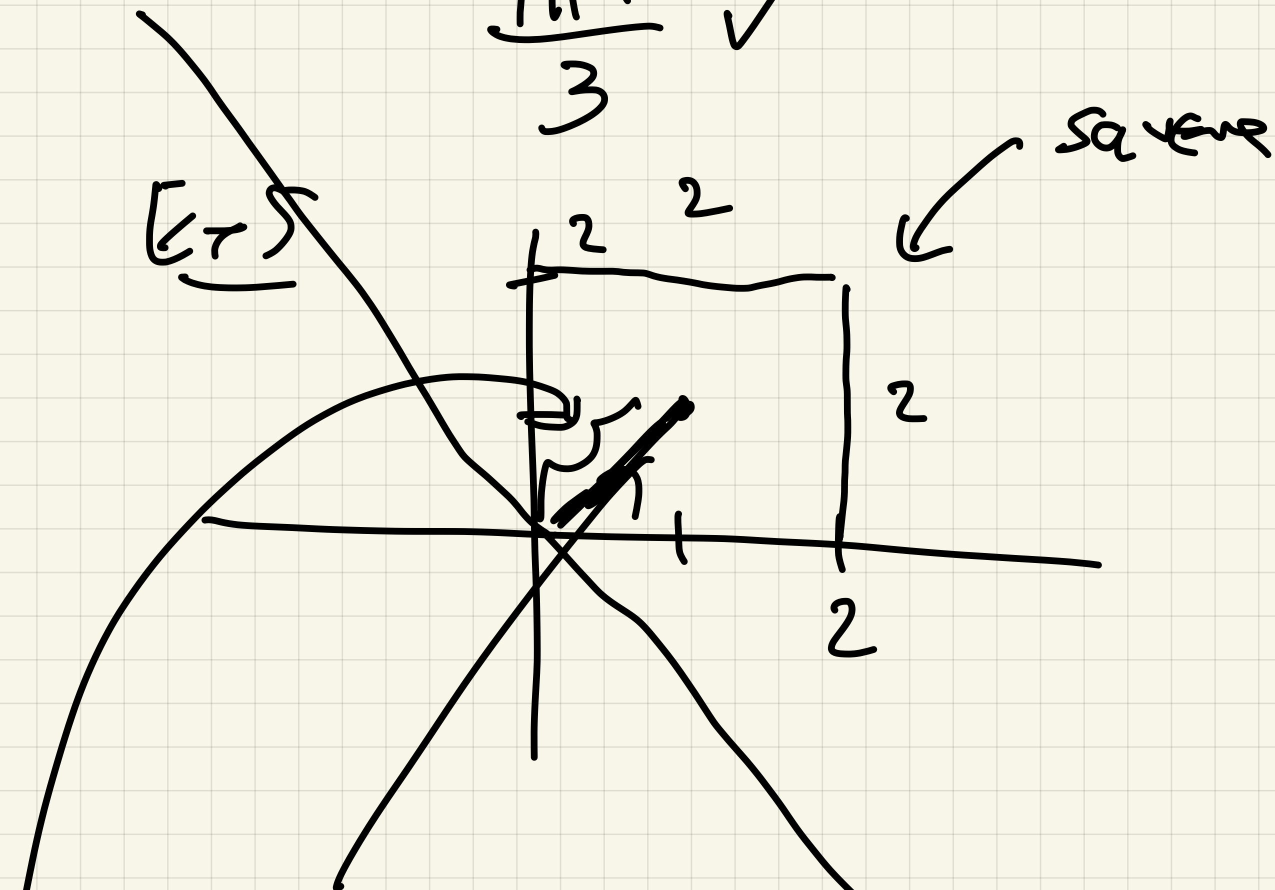
$$\left(0, \frac{2}{3}r^3 \cdot \frac{2}{\pi r^2} = \frac{4r}{3\pi}\right)$$

(b) $r =$ dist to x-axis

is $y = \dfrac{4r}{3\pi} \Longrightarrow$

$$V = 2\pi \bar{r} A = \cancel{2}\pi\left(\frac{4r}{3\cancel{\pi}}\right)\left(\frac{\pi r^2}{\cancel{2}}\right)$$

$$\frac{4\pi r^3}{3} \checkmark$$



$L=$

COM = (1, 1)

$A = 4$

$r = \sqrt{2}$

so $V = 2\pi \cdot \sqrt{2} \cdot 4 = 8\sqrt{2}\pi$