2/16/Calc 2 tank problems

Last time

Strategy

(5) Draw picture

(1) Set variable y for vertical height

(2) Find endpoints a/b corresponding to the setup

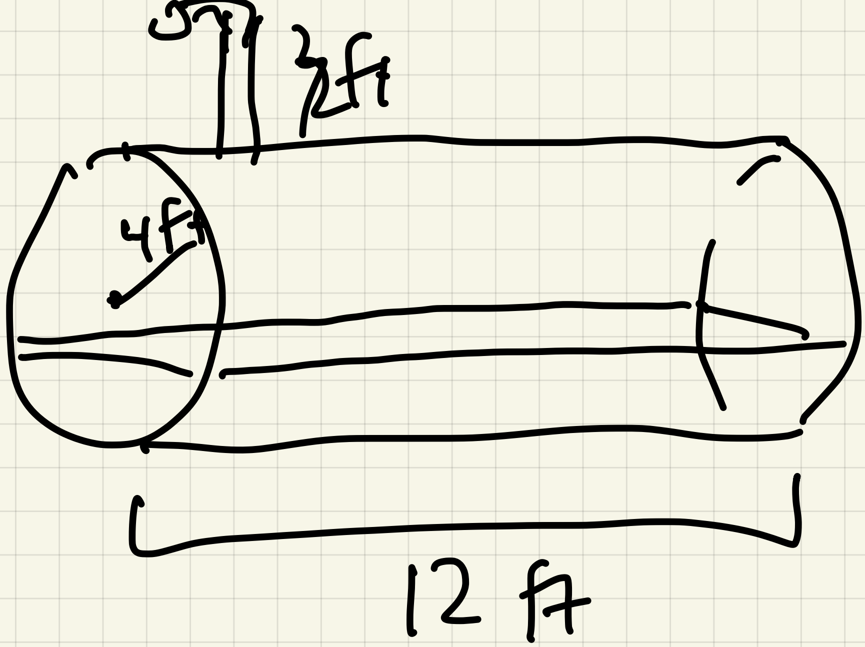
(3) $A(y)$ = cross-sectional area

(4) $D(y)$ = distance moved

$$W = \int_a^b \rho\, A(y)\, D(y)\, dy$$

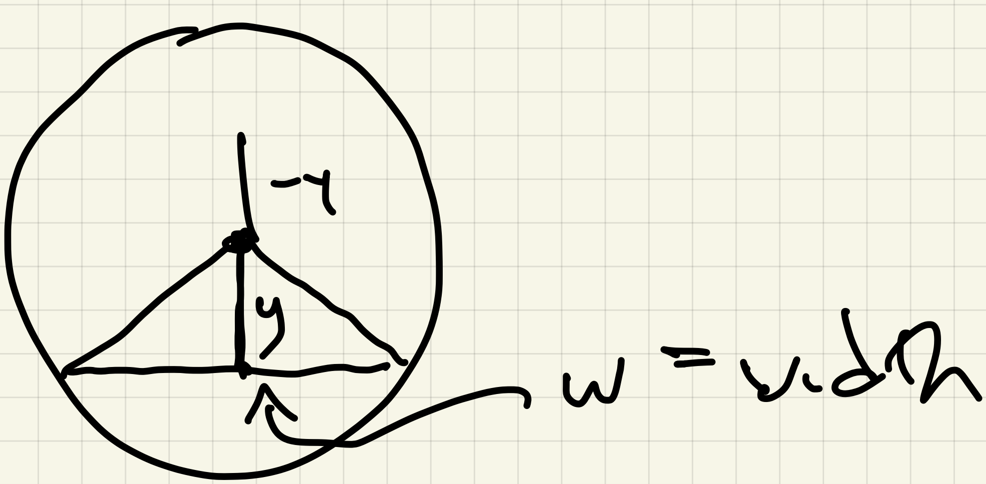
ρ density

Ex1 A cylindrical gas tank, lies on its side as shown:

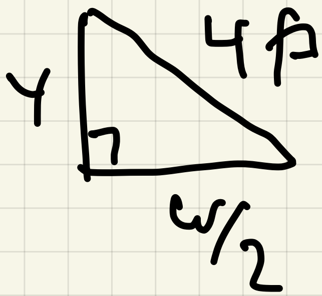


If tank is filled with oil weighing 50 lb/ft^3, How much work is done pumping oil to 2 ft above top of tank?

Need width of cross sections



$y =$ distance below center



$$y^2 + \left(\frac{w}{2}\right)^2 = 4^2 = 16$$

$$\left(\frac{w}{2}\right)^2 = 16 - y^2$$

$$\frac{w}{2} = \sqrt{16 - y^2}$$

$$w = 2\sqrt{16 - y^2}$$

$$A(y) = l \cdot w$$

$$= 12 \cdot 2\sqrt{16 - y^2}$$

$$= 24\sqrt{16 - y^2}$$

end points: $-4 \le y \le 4$

$$D(y) = y + 6$$

$$W = \int_{-4}^{4} \underbrace{(50)\,24}_{ll}\sqrt{16 - y^2}\,(y+6)\,dy$$

$$1200\int_{-4}^{4}(y+6)\sqrt{16-y^2}\,dy$$

$$= 1200\int_{-4}^{4} y\sqrt{16-y^2}\,dy + 1200\int_{-4}^{4} 6\sqrt{16-y^2}\,dy$$

$$1200\int_{-4}^{4} y\sqrt{16-y^2}\,dy = 0$$

$u = 16-y^2$

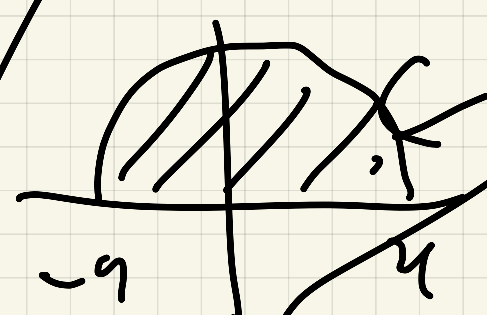
$du = -2y\,dy$

$-\frac{1}{2}du = y\,dy$

$$\int_{u=0}^{u=0}$$

$$1200\int_{-4}^{4} 6\sqrt{16-y^2}\,dy = 7200\int_{-4}^{4}\sqrt{16-y^2}\,dy$$

$$\int_{-4}^{4}\sqrt{16-x^2}\,dx$$



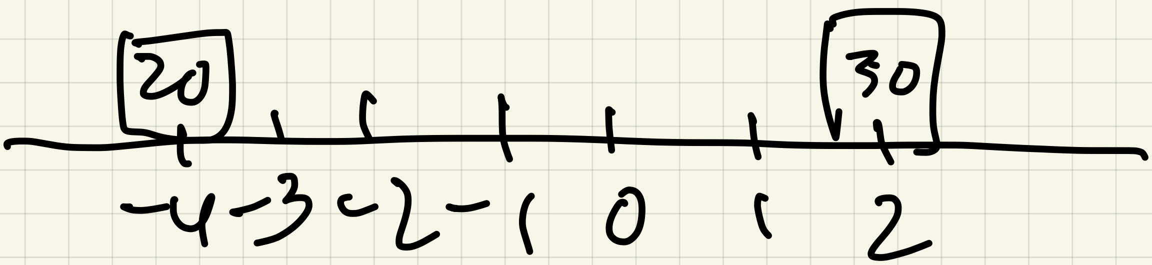
$\text{Area} = \dfrac{\pi \cdot 4^2}{2}$

$7200\,\pi\, 8 = 57{,}600\pi$ ft-lbs

§6.6 Center of mass

Basic idea:

Ex)

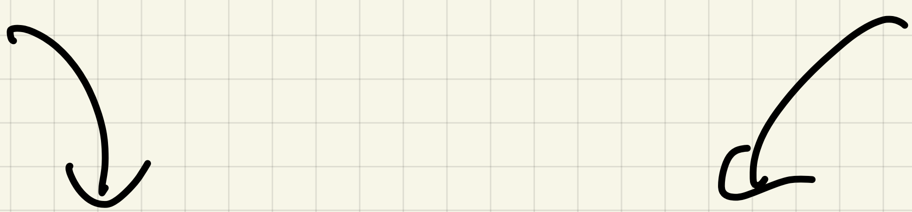


Where is balance point?

Physics: If masses m_i placed at x_i, the moment about y-axis is

$$M_y = \sum m_i x_i$$

measures tendency of system to rotate about y-axis



$M_y > 0$ $M_y < 0$

Total mass: $M = \sum m_i$

Center of mass:

$$\bar{x} = \frac{M_y}{M}$$

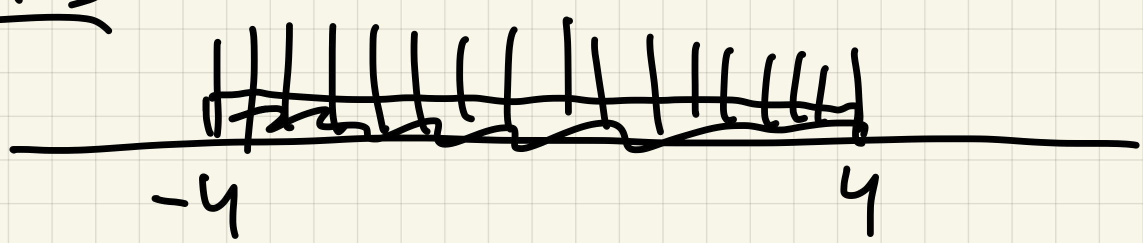
In Ex1: $M_y = -4(20) + 2(30) = -20$

$M = 20 + 30 = 50$

$$\bar{x} = -\frac{20}{50} = -.4$$

What if system continuous instead of discrete?

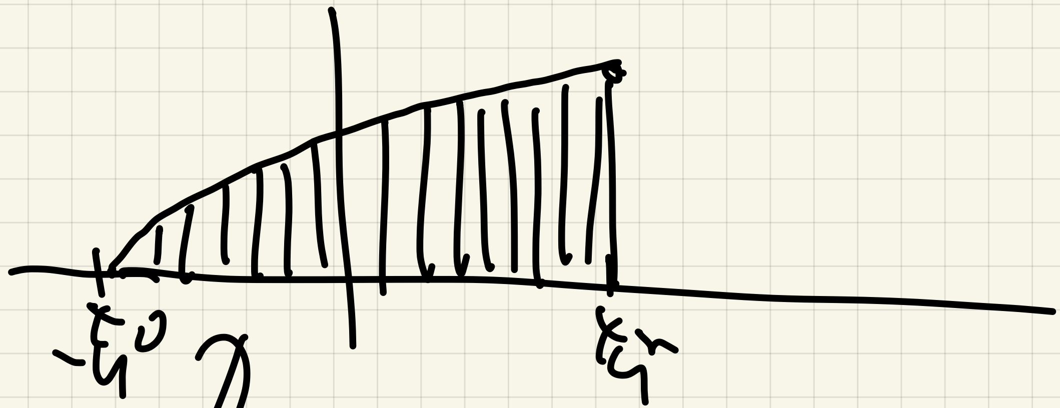
Thin Wires



Ex2

Density at position x is

$$\delta(x) = \sqrt{x+4}$$



Can estimate both M_y and M

divide $[-4,4]$ into n equal pieces

width is $\Delta x = 8/n$

Mass of segment $[x_i, x_{i+1}]$

location x_i

mass $= \sqrt{x_i+4}\,\Delta x = m_i$

Physics: $M_y \approx \sum m_i x_i =$

$$\sum_{i=1}^{n} \left(\sqrt{x_i+4}\right) x_i \Delta x$$

~~estimates~~ $M \approx \sum m_i \approx \sum_{i=1}^{n} \sqrt{x_i+4}\,\Delta x$

Exact values

$$M_y = \int_{-4}^{4} \sqrt{x+4}\cdot x\,dx$$

$$M = \int_{-4}^{4} \sqrt{x+4}\,dx$$

$$M_y = \frac{128}{15}\sqrt{2}$$

$$M = \frac{32\sqrt{2}}{3}$$

$$\Rightarrow \bar{x} = \frac{\frac{128\cancel{\sqrt{2}}}{15}}{\frac{32\cancel{\sqrt{2}}}{3}} =$$

$$\frac{3}{1}\,\frac{4}{15} = \frac{4}{5} = .8$$

In general If thin wire

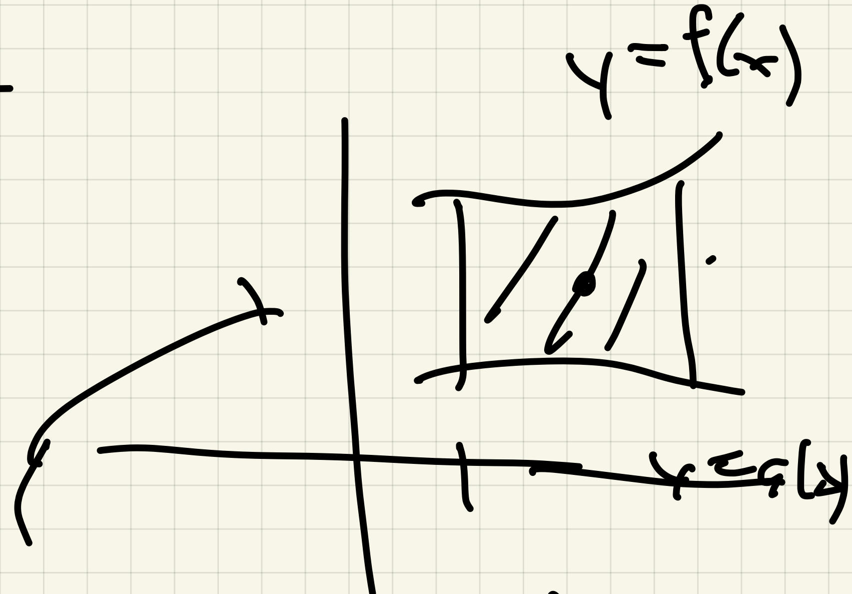
has density $\delta(x)$ at position x

Then $M_y = \int_a^b x\,\delta(x)\,dx$

$$M = \int_a^b f(x)\,dx$$

$$\bar{x} = \frac{M_y}{M}$$

Plates



2-dim problem:

Discrete problem

If masses m_i are placed at positions (x_i, y_i), then

① y-Moment : $M_y = \sum m_i x_i$

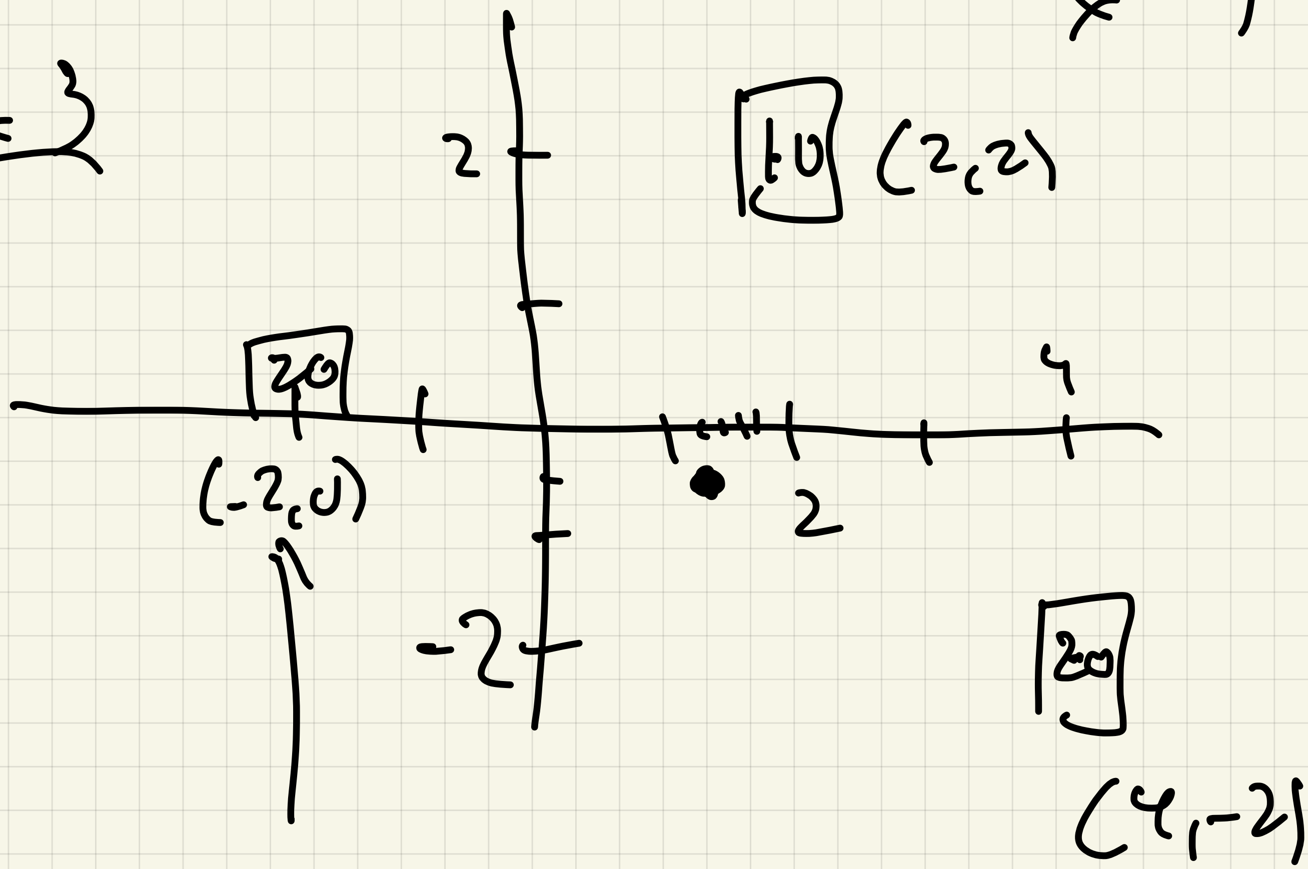
② x-moment $M_x = \sum m_i y_i$

③ Total mass $M = \sum m_i$

Center of mass = COM = $\left(\frac{M_y}{m}, \frac{M_x}{m} \right)$

$$\frac{M_y}{m} = \bar{x}, \quad \frac{M_x}{m} = \bar{y}$$

Ex



$$M_y = -2(20) + 2(10) + 4(20)$$
$$= -40 + 20 + 80 = 60$$

$$M_x = 0(20) + 2(10) + -2(20)$$
$$= 20 - 40 = -20$$

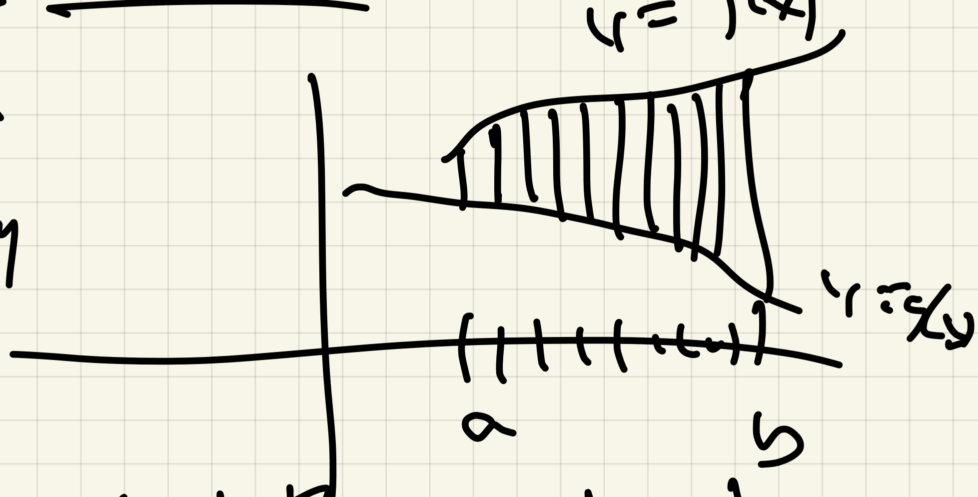
$$m = 50$$

$$\bar{x} = \frac{60}{50} \qquad \bar{y} = \frac{-20}{50}$$

$$= 1.2 \qquad -.4$$

<u>Continuous</u> <u>situation</u>

δ = constant density



m_k = mass of k^{th} rectangle

$$1 \le k \le n \quad \|$$

$$\sum_{k=1}^{n} \delta \, \underbrace{\left(f(x_k) - g(x_k)\right) \Delta x}_{\text{area}}$$

Center of k^{th} rectangle is

$$\left(x_i \; \frac{f(x_k) + g(x_k)}{2}\right)$$

$$\text{so} \qquad \overset{\|}{y_k}$$

$$M_y \approx \sum m_k x_k = \sum \delta\left(f(x_k) - g(x_k)\right) x_k \Delta x$$

$$M_x \approx \sum m_k y_k = \sum \left(f(x_k) - g(x_k)\right)\left(\frac{f(x_k)+g(x_k)}{2}\right)\Delta x$$

$$= \sum \delta \frac{f(x_k)^2 - g(x_k)^2}{2} \Delta x$$

Take limit as $n \to \infty$

$$M = \int_a^b \delta \left(f(x) - g(x)\right) dx$$

$$M_y = \int_a^b \delta \left(f(x) - g(x)\right) x \, dx$$

$$M_x = \int_a^b \delta \frac{f(x)^2 - g(x)^2}{2} dx$$