4/9 | Calc2

Last time

Series tests

nth term

geometric series

integral test

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

DCT $a_n \le b_n$

LCT $\lim \frac{a_n}{b_n} = c \neq 0$

$\sum a_n$ conv $\Leftarrow \sum b_n$ conv.

Absolute converg

$\sum |a_n|$ conv $\Rightarrow \sum a_n$ conv.

Ratio test : $\sum a_n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$$

$r < 1 \Rightarrow \sum a_n$ abs conv

$r > 1 \Rightarrow \sum a_n$ diverges

$r = 1 \Rightarrow$ test fails

Ex 0 $\sum \frac{n^n}{n!}$ $\quad r = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} =$

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$\boxed{= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n =} \quad r$$

$$\ln r = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)^n =$$

$$\lim_{n\to\infty} n \ln\left(\frac{n+1}{n}\right) =$$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}} \overset{LH}{=}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \overset{LH}{=}$$

$$\lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)} \cdot \cancel{-\frac{1}{n^2}}}{\cancel{-\frac{1}{n^2}}} =$$

$$\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1.$$

$\frac{1}{n} \to 0$

Ratio $\therefore r = e^1$

$e > 1 \Rightarrow$ diverges.

<u>But easier</u> : DCT

$$\frac{n^n}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{1} \geq 1$$

$\sum 1$ divergent (geom, $r=1$)

$\therefore$ DCT $\sum \frac{n^n}{n!}$ also divergent

<u>Root Test</u> : $\sum a_n$

Set $r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$. then

① $r < 1 \Rightarrow \sum a_n$ abs. convergent

② $r > 1 \Rightarrow \sum a_n$ divergent

③ $r > 1 \Rightarrow$ test fails

<u>Ex 1</u>

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5^n}$

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{2}{5^n}} =$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{2}}{\sqrt[n]{5^n}} = \lim_{n\to\infty} \frac{2^{1/n}}{5} =$$

$1/n \to 0$

$$\frac{2^0}{5} = \frac{1}{5} < 1 \Rightarrow \text{conv absolutely}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{8}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{8}{n}\right)^n} = \lim_{n\to\infty} \frac{8}{n} = 0 < 1$$

$\therefore$ abs. converges by root test.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

$e > 1 \quad \therefore$ series diverges by root test. by Ex 0

(d) $\sum \frac{1}{n^2}$ (conv by p-series $p = 2 > 1$)

Root test: $r = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} =$

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} \quad \text{"}0^0\text{"}$$

$$\ln r = \lim_{n\to\infty} \ln\left(\frac{1}{n^2}\right)^{1/n} =$$

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{n^2}\right) =$$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{1}{n^2}\right)}{n} \overset{LH}{=}$$

$$\lim_{n\to\infty} \frac{-2\ln n}{n} = \lim_{n\to\infty} \frac{-2/n}{1} = 0$$

$\Rightarrow r = e^0 = 1.$

test fails:

A series is alternating if signs of terms alternate +/−/+.

Ex) $3 - 3/5 + 3/25 - 3/125 + \cdots$

(Geometric, $r = -\frac{1}{5}$, $a = 3$, so converges to $\frac{a}{1-r} =$

$$\frac{3}{1-\left(-\frac{1}{5}\right)} = \frac{3}{(6/5)} = \frac{15}{6} = \frac{5}{2}$$)

Note: If $\sum b_n$ alternates, can pull out sign ~~as~~

$(-1)^n$ $(-1)^{n+1}$

write

$$\sum b_n = \begin{cases} \sum (-1)^n a_n \\ \sum (-1)^{n+1} a_n \end{cases} \quad a_n \geq 0$$

Alternating series test (AST)

1. $\lim a_n = 0$
2. a_n decrease

$\Rightarrow \sum (-1)^n a_n$ convergent.

Ex: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \ldots$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\frac{1}{n+1} < \frac{1}{n}$ ✓

$\Rightarrow$ series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 \ldots$

diverges (by geometric series)

diverges n^{th} term test

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots$

$\lim \frac{1}{n!} = 0$ ✓

$\frac{1}{(n+1)!} < \frac{1}{n!}$ ✓ $\Rightarrow$ AST converges

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n n^{60}}{e^n}$

① $\lim_{n\to\infty} \frac{n^{60}}{e^n} \overset{\text{LH} \times 60}{=} \lim_{n\to\infty} \frac{60!}{e^n} = 0$ ✓

② decreasing $\frac{n^{60}}{e^n}$

$f(x) = \frac{x^{60}}{e^x}$ $\quad f'(x) < 0$ $\quad x > 60$

so decreases for $n \geq 60$

$\sum_{n=60}^{\infty} (-1)^n \frac{n^{60}}{e^n}$ converges by AST

$\Downarrow$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{60}}{e^n} \quad \text{converges}$$

Ex] We've seen

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad -1<r<1$$

If we think r as a variable x

then $(a=1)$

$$\sum r^k = \frac{1}{1-r}$$

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$= 1+x+x^2+x^3+\dots$$

Defn: If x is a variable, then a power series is an infinite series of form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

coefficients

A <u>power series centered at c</u> has form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

<u>Defn</u>: The <u>Interval of convergence</u> (IOC) is $\{x : f(x) \text{ converges}\}$

<u>Ex1</u> $f(x) = \sum_{n=0}^{\infty} \frac{1}{5^n}(x-4)^n =$

$$1 + \frac{1}{5}(x-4) + \frac{1}{25}(x-4)^2 + \dots$$

Center $c = 4$ coeffs $a_n = \frac{1}{5^n}$

When does it converge?

Ratio test:

$$\lim \left| \frac{\frac{1}{5^{n+1}} (x-4)^{n+1}}{\frac{1}{5} (x-4)^n} \right| = \lim \frac{1}{5}(x-4)$$

conv. if

$$-1 < \frac{1}{5}(x-4) < 1$$