4/7/ Calc2

DCT $0 \le a_n \le b_n$

Last time ① $\sum b_n$ converges

$\Downarrow$

$\sum a_n$ converges

② $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

LCT $0 < a_n, b_n$

① $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = c \neq 0 \Rightarrow$

$\sum a_n, \sum b_n$ — both diverge

or

both converge

② $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$

$\sum b_n$ converges $\Rightarrow \sum a_n$ converges

③ $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = +\infty$,

$\sum b_n$ diverge $\Rightarrow \sum a_n$ diverges

Ex (a) $\sum\limits_{n=2}^{\infty} \frac{12}{\sqrt[8]{n^9 - 17}} \sim \sum \frac{1}{\sqrt[8]{n^9}}$

$$\lim_{n\to\infty} \frac{\frac{12}{\sqrt[8]{n^9-17}}}{\frac{1}{\sqrt[8]{n^9}}} = \lim_{n\to\infty} 12 \frac{\sqrt[8]{n^9}}{\sqrt[8]{n^9-17}}$$

$$= \lim_{n\to\infty} 12 \sqrt[8]{\frac{n^9}{n^9-17}} = 12\sqrt[8]{1} = 12$$

$$\lim \frac{9n^8}{9n^8} = \lim 1 = 1 \checkmark$$

LH

by LCT, $\sum \frac{12}{\sqrt[8]{n^9-17}}$ converges

(b) $$\sum_{n=2}^{\infty} \frac{\sqrt[7]{3n^3+1}}{\sqrt[2]{7n^3-2}} \sim \frac{n^{3/7}}{n^{3/2}} =$$

$$\frac{1}{n^{3/2-3/7}} = \frac{1}{n^{15/14}} \quad \text{converges}$$

So far, tests deal series $\sum a_n$, $a_n \geq 0$ (LCT, DCT, Int test)

Often this is enough:

<u>Thm 1</u> $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges

<u>Ex 1</u> (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} - \cdots$

$$\sum \left| \frac{(-1)^n}{n^3} \right| = \sum \frac{1}{n^3}$$ converges by p-series $p = 3 > 1$ ✓

$\xrightarrow{\text{Thm 1}}$ $\displaystyle\sum \frac{(-1)^n}{n^3}$ converges.

<u>Defn</u>: A series $\sum a_n$ is <u>absolutely</u> <u>convergent</u> if

$\sum |a_n|$ converges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$n! = \begin{cases} 1 & n = 0 \\ n(n-1)(n-2)\ldots(1) & n > 0 \end{cases}$$

Check if $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

DCT: $\displaystyle\frac{1}{n(n-1)(n-2)\ldots(1)} \le \frac{1}{n(n-1)}$

$$\frac{1}{n(n-1)} \le \frac{1}{(n-1)^2}$$

$$\sum_{n=3}^{\infty} \frac{1}{(n-1)^2} = \underbrace{\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots}_{p\text{-series},}$$

converges b/c $p = 2 > 1$

So DCT $\sum \frac{1}{n!}$ conve

Thm 1 $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges $= \frac{1}{e}$

argument clumsy!

Thm 2 (Ratio Test)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$$

① $r<1 \Rightarrow \sum a_n$ absolutely convergent

② $r>1 \Rightarrow \sum a_n$ divergent

③ $r=1 \Rightarrow$ test fails.

Try it: on $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$$r = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} =$$

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n}\cancel{(n-1)}\cdots \cancel{1}}{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{1}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$\therefore \sum \frac{(-1)^n}{n!}$ absolutely convergent.

<u>Ex 3</u> (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$ (Geom)

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}}{3^{n+1}}}{\frac{(-2)^n}{3^n}} \right| = \lim \frac{2}{3} = \frac{2}{3} < 1$$

$\therefore$ abs convergent.

(b) $$\sum_{n=1}^{\infty} \frac{n!}{100^n} = \frac{1}{100} + \frac{2}{10000} + \frac{6}{10^6} + \frac{24}{10^8} + \cdots$$

Ratio test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \lim_{n\to\infty} \frac{(n+1)}{100} = \infty$$

$\checkmark$

so diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{60}}{e^n}$

Apply Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{60}}{e^{n+1}}}{\frac{n^{60}}{e^n}} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{60} \cdot \frac{1}{e}$$

1

$$\frac{e^n}{e^{n+1}} = \frac{1}{e}$$

$= \frac{1}{e} < 1$ $\therefore$ series converges

(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} =$$

$$\lim_{n\to\infty} \left(\boxed{\frac{(n+1)!}{n!}}\right)^2 \quad \frac{(2n)!}{(2n+2)!}$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{1} \quad \frac{2n(2n-1)(2n-2)\cdots(1)}{\boxed{(2n+2)(2n+1)}(2n)(2n-1)\cdots(1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{4n+2} = \frac{1}{4} < 1 \rightarrow$$

convergent

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n-1}}{n^2}$

<u>Ratio test</u>: $\displaystyle\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{(n+1)^2}}{\frac{\sqrt{n-1}}{n^2}} =$

$$\lim_{n\to\infty} \sqrt{\frac{n}{n-1}} \cdot \left(\frac{n}{n+1}\right)^2 = \sqrt{1} \cdot 1^2 = 1$$

<u>Test fails</u>.

<u>BCT</u>: DCT $\quad \dfrac{\sqrt{n-1}}{n^2} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$

conv. p-series $p = 3/2 > 1$

$\therefore \sum \dfrac{\sqrt{n-1}}{n^2}$ converges by DCT.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n \to \infty} \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \lim_{n \to \infty} \underbrace{\frac{n!}{(n+1)!}}_{\frac{1}{(n+1)}} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \lim_{n \to \infty} \frac{(n+1)^n}{n^n} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n$$