4/24/Calc 2 Exam 3 $\to$ Tuesday

Taylor polynomials

$f(x) \to \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$

$c$

partial sums

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = \lim_{n\to\infty} P_n(x)$$

Taylor's Formula/Thm

$$f(b) = P_n(b) + R_n(b)$$

$$R_n(b) = \frac{f^{(n+1)}(z)}{(n+1)!}(b-c)^{n+1}$$

some $z$ between $b$ & $c$.

Ex 0 $\sin x = f(x)$

$P_5(x)$

$\sin 1 \approx P_5(1)$

$= 1 - \frac{1}{3!} + \frac{1}{5!}$

$|R_n(1)| < .0013\overline{8}$

Estimate $\sqrt{101}$

Ex 1 $\quad f(x) = \sqrt{x}, \quad c = 100$

We computed

$$P_2(x) = 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2$$

$$\sqrt{101} = f(101) \approx P_2(101) =$$

$$10 + \frac{1}{20} - \frac{1}{8000} =$$

$$\boxed{10.049875}$$

error $\quad R_2(101) = \underbrace{\frac{f^{(3)}(z)}{3!}(101-100)^3}$

$100 \le z \le 101$

$$\left| \frac{f^{(3)}(z)}{3!}(101-100)^3 \right|$$

$$f = x^{1/2} \quad f' = \frac{1}{2}x^{-1/2}, \quad f'' = -\frac{1}{4}x^{-3/2}$$

$$f''' = \frac{3}{8} x^{-5/2} =$$

$$\left| \frac{\frac{3}{8(z)^{5/2}}}{3!} \right| \le \frac{\frac{3}{8\cdot(100)^{5/2}}}{3!} =$$

$$100 \le z \le 101$$

$$\frac{\cancel{3}}{2!\cdot\cancel{3}\cdot 8\cdot} \quad 100^{5/2} = 10^5$$

$$100^{1/2} = 10$$

$$\frac{1}{16\times 10^5} = \underline{\underline{.00000625}}$$

**Note:**

calculator

$$\sqrt{101} = \boxed{10.049875}6211$$

**Question**

Is $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ ??

( For $x$ in interval of convergence

$$\lim_{n\to\infty} P_n(x)$$

Is $f(x) = \lim_{n\to\infty} P_n(x)$

Is

$$0 = f(x) - \lim_{n\to\infty} P_n(x)$$

$$= \lim_{n\to\infty} \underbrace{f(x) - P_n(x)}$$

$$= \lim_{n\to\infty} R_n(x) = 0 \text{ ??}$$

<u>Ex 1</u> (a) Compute Maclaurin series for $\cos x$

(b) find I.O.C.

(c) Show $f(x) =$ Maclaurin series

(a)

(a)

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)/k!$ |
|---|---|---|
| 0 | $\cos x$ | 1 |
| 1 | $-\sin x$ | 0 |
| 2 | $-\cos x$ | $-1/2!$ |
| 3 | $\sin x$ | 0 |
| 4 | $\cos x$ | $1/4!$ |
| 5 | $-\sin x$ | 0 |
| 6 | $-\cos x$ | $-1/6!$ |
| | $\vdots$ | |

Series $$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

(b) Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} x^{2n+2}}{(2n+2)!}}{\frac{\cancel{(-1)^n} x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} x^2 \frac{(2n)!}{(2n+2)!}$$

$$\lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

$$I.o.C = (-\infty, \infty) = \mathbb{R}$$

(c) For exactility, need to show

$$\lim_{n\to\infty} R_n(x) = 0$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-0)^{n+1} \right| \leq$$

$$|\pm \cos z / \pm \sin z| \leq 1$$

$$\left| \frac{x^{n+1}}{(n+1)!} \right|$$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$ convergent series by Ratio test

$\Downarrow$ n<sup></sup>th term test

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

<u>Conclusion</u>:

all $x$

$$\begin{cases} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{cases}$$

$$\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$-1 < x < 1$

<u>Rmk</u>: If $f(x) = \sum a_n x^n$

then $a_n = \frac{f^{(n)}(0)}{n!}$

## Ex 2: Manipulation of Taylor series

(a) $\dfrac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k$

$\downarrow$

Differentiate

(b) $\dfrac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots$

$= \sum_{k=1}^{\infty} k x^{k-1}$

(c) $-\ln|1-x| =$

Integrate

$(C+)\ x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots$

but

$C = 0$

$= \sum_{k=0}^{\infty} \dfrac{x^{k+1}}{k+1}$

$-1 < x < 1$

Exa. $x = \dfrac{1}{2}$

$-\ln\dfrac{1}{2} = \dfrac{1}{2} + \dfrac{\left(\frac{1}{2}\right)^2}{2} + \dfrac{\left(\frac{1}{2}\right)^3}{3} + \ldots$

$\|$ $\qquad$ $\|$

$$\ln 2 = \frac{1}{2} + \frac{1}{2^2\cdot 2} + \frac{1}{2^3\cdot 3} + \frac{1}{2^4\cdot 4} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{k\cdot 2^k}$$

$\| $

$.69314\ldots$

(d) subst $x^2$ for $x$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$

(e) subst $-x^2$ for $x$.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

(f) Integrate (e)

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

$x = 1$

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

$$\shortparallel$$

$$\pi/4$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

(g) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 - -$$

subst $x$ with $-x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + -$$

(h) Add those:

$$\frac{1}{1-x} + \frac{1}{1+x} = 2 + 2x^2 + 2x^4 \dots$$

$$\shortparallel$$

$$\frac{2}{1-x^2}$$

Ex 1 Find Maclaurin series for $x^3 e^{-x} = f(x)$

What is $f^{(100)}(0)$?

$f' = 3x^2 e^{-x} - x^3 e^{-x}$

$f'' = 6x e^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x}$

$\vdots$

bad idea

other approach:

Manipulate

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\Downarrow$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

mult by $x^3$

$x^3 e^{-x} = x^3 - x^4 + \frac{x^5}{2!} - \frac{x^6}{3!} + \cdots$

$$- - - \cdots - \frac{x^{100}}{97!}$$

$$\frac{f^{(100)}(0)}{100!} = \frac{-1}{97!}$$