4/14/Calc2

Series tests

Last time

Power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

coefs $\quad$ center

I.O.C. = $\{x : f(x) \text{ converges}\}$

<u>Ex1</u> (a) $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for <u>all</u> x by Ratio test

I.O.C. $= (-\infty, \infty)$

(b) $\sum_{n=0}^{\infty} \frac{(x+1)^n}{3^n \sqrt{n+10}}$

<u>Ratio test</u>: $r = \lim \left| \dfrac{\dfrac{(x+1)^{n+1}}{3^{n+1}\sqrt{n+11}}}{\dfrac{(x+1)^n}{3^n \sqrt{n+10}}} \right| =$

$$\frac{|x+1|}{3} < 1 \Rightarrow \text{converges absolutely}$$

$$\frac{|x+1|}{3} > 1 \Rightarrow \text{diverges}$$

$$\frac{|x+1|}{3} = 1 \Rightarrow \text{test fails}$$

$$|x+1| < 3 \Leftrightarrow -3 < x+1 < 3$$

$$\Leftrightarrow -4 < x < 2$$

<u>Check endpoints</u>:

<u>$x = 2$</u>: $\displaystyle\sum \frac{(x+1)^n}{3^n\sqrt{n+10}} = \sum \frac{3^n}{3^n\sqrt{n+10}} =$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+10}} \sim \frac{1}{\sqrt{n}}$$

LCT $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+10}}}{\frac{1}{\sqrt{n}}} =$

$$\lim_{n\to\infty} \sqrt{\frac{n}{n+10}} = 1 \neq 0$$

$\sum \frac{1}{\sqrt{n}}$ divergent b/c p-series $p = \frac{1}{2} \leq 1$

LCT $\Rightarrow \sum \frac{1}{\sqrt{n+10}}$ diverges

<u>$x = -4$</u>

$$\sum \frac{(-3)^n}{\cancel{3^n}\sqrt{n+10}} = \cancel{3^n}(-1)^n$$

$$= \sum \frac{(-1)^n}{\sqrt{n+10}}$$

$\lim \frac{1}{\sqrt{n+10}} = 0$ ✓ $\therefore$ conv by AST

$\frac{1}{\sqrt{n+10}}$ decreases ✓

$\therefore$ I.O.C. $[-4, 2)$

General Picture:

A power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges in one of 3 ways:

(A) There is a number $R > 0$:
$f(x)$ conv for $|x-c| < R$
diverges $|x-c| > R$

(B) $f(x)$ converges for all x $(R = \infty)$

(C) $f(x)$ converges ONLY for $x = c$ $(R = 0)$

$R =$ radius of convergence

Ex1(a) $R = \infty$

Ex1(b) $R = 3$

Ex2 $\sum_{n=0}^{\infty} n!\,(x-7)^n =$
(a) $1 + (x-7) + 2(x-7)^2 +$

$6(x-7)^3 + \ldots$

Ratio test:

(a) $$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-7)^{n+1}}{n!\,(x-7)^n} \right| =$$

$$\lim_{n\to\infty} \underbrace{(n+1)}_{\infty} |x-7| = \begin{cases} \infty & \text{if } x \neq 7 \\ 0 & x = 7 \end{cases}$$

(b) $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} =$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim \frac{|x^2|}{(2n+3)(2n+2)} = 0$$

$(2n+1)!$

$$\frac{(2n+1)!}{(2n+3)!}$$

$\therefore$ I.O.C $= (-\infty, \infty) = \mathbb{R}$

<u>Thm</u> $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, R = radius of convergence,

f(x) converges on $(c-R, c+R)$

① $f'(x) = \sum_{n=1}^{\infty} n \cdot a_n (x-c)^{n-1}$

② $\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

AND radius of convergence is R

<u>Ex 2</u> $f(x) = \sum_{n=1}^{\infty} \frac{(x-10)^n}{n \cdot 7^n}$

$$\lim_{n\to\infty}\left|\frac{\frac{(x-10)^{n+1}}{(n+1)\,7^{n+1}}}{\frac{(x-10)^n}{n\cdot 7^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-10)}{7}\frac{n}{n+1}\right|$$

$$= \frac{|x-10|}{7} < 1$$

$$|x-10| < 7$$

$$-7 < x-10 < 7$$

$$3 < x < 17 \qquad \boxed{R=7}$$

endpoints :

$$\sum_{n=1}^{\infty}\frac{(x-10)^n}{n\cdot 7^n} : \quad \boxed{x=17}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n} = \text{diverges}$$

$\boxed{x=3}$

$$\sum\frac{(-1)^n}{n} \quad \text{AST converges}$$

So $[3, 17)$

Ex 3 a Find $f'(x)$, $\int f(x)$,

and interval of convergence

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-10)^n}{n \cdot 7^n}$$

(a) $$f'(x) = \sum_{n=1}^{\infty} \frac{\cancel{n}(x-10)^{n-1}}{\cancel{n} \cdot 7^n} =$$

$$\sum_{n=1}^{\infty} \frac{(x-10)^{n-1}}{7^n} \left(= \sum_{m=0}^{\infty} \frac{(x-10)^m}{7^{m+1}} \right)$$

Endpoints: $x = 3$, $x = 17$

geometric, $r = \frac{x-10}{7}$

conv.: $-1 < \frac{x-10}{7} < 1$

$3 < x < 17$

I.O.C. $(3, 17)$

(b) $f(x) = \sum_{n=1}^{\infty} \frac{(x-10)^n}{\underbrace{n \cdot 7^n}}$

$$\int f(x) = C + \sum_{n=1}^{\infty} \frac{(x-10)^{n+1}}{n(n+1)\, 7^n}$$

<u>Check endpoints:</u>

$x = 17$: $\quad \sum \frac{7^{n+1}}{n(n+1)\, 7^n} =$

$7 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leftarrow$ converges:

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ converges $\left(\begin{array}{l} p = 2 > 1 \\ p\text{-series} \end{array} \right)$

$\therefore \cancel{7} \sum_{n} \frac{7^{n+1}}{n(n+1)\, 7^n}$ converges by DCT.

$\underline{x=3}$

$$\sum \frac{(-7)^{n+1}}{n(n+1)7^n}$$

converges b/c

$$\sum \left| \quad \right| = \sum \frac{7^{n+1}}{n(n+1)7^n}$$

converges.

$$[3, 17]$$

<u>Follow up on Ex 3:</u>

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-10)^{n-1}}{7^n} \qquad (3, 17)$$

$$= \frac{1}{7} + \frac{(x-10)}{7^2} + \frac{(x-10)^2}{7^3}$$ geom, converges

to $\frac{a}{1-r} = \frac{\frac{1}{7}}{1 - \frac{(x-10)}{7}} =$

$$\frac{1}{7-(x-10)} = \frac{1}{17-x}$$

So $f'(x) = \dfrac{1}{17-x}$

$$f(x) = \int f'(x)\,dx = \int \frac{1}{17-x} =$$

$$f(x) = -\ln|17-x| + \underline{\underline{C}}$$

$f(10) = 0$, $\therefore$

$$0 = f(10) = -\ln 7 + C$$

$$C = \ln 7 \quad \therefore$$

$$f(x) = \ln 7 - \ln|17-x|$$

§ 9.8 ?

Defn If $f(x)$ is a function, the <u>n^{th} Taylor polynomial of $f(x)$ centered at c</u> is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Defn The <u>Taylor series for $f(x)$ at $x=c$</u> is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Ex

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$P_n(c) = f(c)$$

$$P_n'(c) = f'(c)$$

$$P_2''(c) = f''(c)$$

Remark: By construction,

$$f^{(k)}(c) = P_n^{(k)}(x)$$

Ex 1 $\quad f(x) = x^3 \qquad c=2$

k	$f^{k}(x)$	$f^{(k)}(2)/k!$
0	x^3	8
1	$3x^2$	12
2	$6x$	$12/2$
3	6	$6/3! = 1$
4	0	0
5	0	0

So Taylor series is

$$8 + 12(x-2) + 6(x-2)^2 + 1(x-2)^3$$