NOETHER-LEFSCHETZ THEORY AND QUESTIONS OF SRINIVAS

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Abstract. Noether-Lefschetz theory surrounds the question of determining when the restriction map from the Picard group of a variety to the Picard group of a general member of a linear system is an isomorphism, its origins dating back to Lefschetz' proof from 1921 of a claim made my Noether in 1882. After discussing the Grothendieck-Lefschetz theorems in higher dimensions, we focus on modern results in the three-dimensional case, finishing with a new proof of Moishezon's theorem. Then we will discuss problems in local commutative algebra posed by Srinivas [38] and recent results obtained from Noether-Lefschetz theory.

1. Noether-Lefschetz theory

Modern Noether-Lefschetz theory surrounds the following problem:

Problem 1.1. For which complex varieties X and line bundles $L \in \operatorname{Pic} X$ is the restriction map $r_Y : \operatorname{Pic} X \to \operatorname{Pic} Y$ an isomorphism for general Y in the linear system $|L| = \mathbb{P}H^0(L)$?

For pairs (X, L) as bove, the following question has also received plenty of attention:

Problem 1.2. Describe the irreducible families $V \subset |L|$ of surfaces for which the restriction map r_Y fails to be an isomorphism: these are the *Noether-Lefschetz components*.

I will discuss results over $\mathbb{C}$, but in their study of monodromy groups in characteristic $p > 0$ [11, 19], Grothendieck, Deligne and Katz extended Lefschetz pencils, vanishing cycles and the Picard-Lefschetz formula to obtain results on Problem 1.1 meaningful in finite characteristic.

1.1. The Noether-Lefschetz theorem.

The statement is as follows:

Theorem 1.3. *For $d > 3$, the restriction map* $\operatorname{Pic} \mathbb{P}^3 \to \operatorname{Pic} Y$ *is an isomorphism for very general $Y \in |H^0(\mathcal{O}_{\mathbb{P}^3}(d))|$.*

1.1.1. *Noether's idea.* Count dimensions.

Example 1.4. Let $V \subset |\mathcal{O}(d)|$ be the family of surfaces Y containing a line L. Let

$$I = \{(L, S) : L \subset S\} \subset \mathbb{G}(1, 3) \times V$$

be the incidence variety of lines on surfaces along with projections $\pi_2 : I \to V$ and $\pi_1 : I \to \mathbb{G}(1, 3)$, where $\mathbb{G}(1, 3)$ is the Grassmann variety of lines. The exact sequence

$$0 \to H^0(\mathcal{I}_L(d)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_L(d)) \to 0$$

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shows that $\dim \pi_1^{-1}(L) = \dim |\mathcal{O}(d)| - d - 1$, hence $\dim I \leq \dim |\mathcal{O}(d)| - d + 3$ since $\dim \mathbb{G}(1,3) = 4$. Therefore $\dim V \leq \dim |\mathcal{O}(d)| - d + 3$ so that $V \subset |\mathcal{O}(d)|$ is a proper subvariety for $d > 3$.

Apparently Noether did many such calculations, leading to his conclusion.

Exercise 1.5. Carry out Noether's dimension count for surfaces containing conics.

1.1.2. *Lefschetz' proof.* Lefschetz easily proved an analogous statement in higher dimensions. The exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$ yields

$$\begin{array}{ccccccc} H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) & \to & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*) & \to & H^2(\mathbb{P}^n, \mathbb{Z}) & \to & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \\ \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ H^1(Y, \mathcal{O}_Y) & \to & H^1(Y, \mathcal{O}_Y^*) & \to & H^2(Y, \mathbb{Z}) & \to & H^2(Y, \mathcal{O}_Y). \end{array} \tag{1}$$

For $n > 3$ the cohomology groups in the four corners are zero and α is identified with the restriction map $\operatorname{Pic} \mathbb{P}^n \to \operatorname{Pic} Y$. The Lefschetz hyperplane theorem says that the maps $H^k(\mathbb{P}^n, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ are isomorphisms for $k < n-1$ and injective for $k = n-1$. Thus β is an isomorphism for $n > 3$ and therefore α as well:

Theorem 1.6. *If $Y \subset \mathbb{P}^n$ is a smooth hypersurface and $n > 3$, then the restriction map $\operatorname{Pic} \mathbb{P}^n \to \operatorname{Pic} Y$ is an isomorphism.*

When $n = 3$ the result no longer follows from Diagram (1) because there is no zero in the lower right and the Lefschetz hyperplane theorem no longer applies to β. The statement fails for $d = 2$ and $d = 3$ and for $d > 3$ there are infinitely many Noether-Lefschetz components $V \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$, so the restriction map is not an isomorphism for Zariski general $Y \in |\mathcal{O}_{\mathbb{P}^3}(d)|$. Lefschetz used a monodromy argument, showing that a typical deformation (along a Lefschetz pencil) takes a Hodge class $\gamma \in H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{Z})$ representing a non-complete intersection curve into $H^{0,2}(Y, \mathbb{C})$ and therefore becomes non-algebraic. Voisin gives a clear exposition in her books on Hodge theory [41, 42].

Remark 1.7. Mumford's challenge from the 1960s to find an explicit equation of a smooth quartic $S \subset \mathbb{P}^3$ with $\operatorname{Pic} S = \langle \mathcal{O}_S(1) \rangle$ was finally answered by van Luijk in 2007. One such equation [40, Remark 3.7] is

$$w(x^3 + y^3 + x^2z + xw^2) = 3x^2y^2 - 4x^2yz + x^2z^2 + xy^2z + xyz^2 - y^2z^2.$$

Note that this surface contains the line $w = z = 0$ in characteristic $p = 3$, so this surface specializes to a member of the Noether-Lefschetz locus in finite characteristic.

1.2. **Higher dimension: Grothendieck-Lefschetz theorems.** When $\dim X > 3$, the results are excellent.

Theorem 1.8. *Let X be a smooth projective variety of dimension $n \geq 4$. Then for* any *effective ample divisor $Y \subset X$, the restriction map $\operatorname{Pic} X \to \operatorname{Pic} Y$ is an isomorphism.*

For example, *every* closed subscheme $Y \subset \mathbb{P}^4$ defined by a homogeneous polynomial has Picard group $\operatorname{Pic} Y$ generated by $\mathcal{O}_Y(1)$. Hartshorne [21, IV, Corollary 3.3] simplified Grothendieck's original proof [18] by assuming X and Y nonsingular, but Lazarsfeld observes that the smoothness of Y was unncessary [25, Remark 3.1.26]. Grothendieck's original idea [18, Exposé X] is to consider an open neighborhood U of Y in the formal completion $\hat{X}$ of X along Y and show that the sequence of induced maps

$$\operatorname{Pic} X \to \operatorname{Pic} U \to \operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$$

are all isomorphisms. The most difficult part is the isomorphism $\operatorname{Pic} \hat{X} \cong \operatorname{Pic} U$, for which Grothendieck defines *effective Lefschetz conditions* $\operatorname{Leff}(X, Y)$ that are satisfied by the pair (X, Y). The last isomorphism is obtained by considering the infinitesimal neighborhoods $Y_n \subset X$ defined by ideals $\mathcal{I}_Y^n$. Kodaira vanishing implies that $H^i(Y, \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}) = 0$ for $i = 1, 2$ and therefore the exact sequences $0 \to \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1} \to \mathcal{O}_{Y_n}^* \to \mathcal{O}_{Y_{n+1}}^* \to 0$ give isomorphisms $\operatorname{Pic} Y_n \cong \operatorname{Pic} Y_{n+1}$ for $n > 0$ and hence $\operatorname{Pic} \hat{X} \cong \varprojlim \operatorname{Pic} Y_n \cong \operatorname{Pic} Y$.

1.2.1. *Normal varieties and class groups.* For X normal, one can consider the strict transform $\tilde{Y} \subset \tilde{X}$ for a desingularization $\tilde{X} \to X$. If $E \subset \tilde{X}$ is the exceptional divisor, the for general member $Y \subset X$ of a base point free linear system we have the homomorphism $\operatorname{Cl} X \cong \operatorname{Pic}(X - E) \to \operatorname{Pic}(Y - E) \cong \operatorname{Cl} Y$. Ravindra and Srinivas [34] prove an "almost" Lefschetz condition $\operatorname{ALeff}(\tilde{X}, \tilde{Y})$ which leads to the following analog for class groups.

Theorem 1.9. *Let X be a normal variety of dimension $n \geq 4$. Assume $L \in \operatorname{Pic} X$ is ample and $V \subset H^0(X, L)$ is a base point free linear system. Then the general member $Y \in |V|$ is normal and the restriction map $r : \operatorname{Cl} X \to \operatorname{Cl} Y$ is an isomorphism.*

1.2.2. *Linear systems with base locus.* We used Theorem 1.9 in our result for linear systems with base locus [4].

Theorem 1.10. *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a normal variety of dimension $n \geq 4$ and let $Z \subset X$ be a closed subscheme of codimension ≥ 2 with codimension 2 irreducible components $Z_1, Z_2, \ldots, Z_s$. Assume*

1. *$\mathcal{I}_Z(d-1)$ is generated by global sections.*
2. *Z_i is not contained in the singular locus of X.*
3. *Z_i has generic embedding dimension at most $\dim X - 1$.*

Then the general member $Y \in |H^0(X, \mathcal{I}_Z(d))|$ is normal and the map $\alpha : \operatorname{Cl} X \oplus \mathbb{Z}^s \to \operatorname{Cl} Y$ given by $(L, a_1, \ldots, a_s) \mapsto L|_Y + \sum a_i \operatorname{Supp} Y_i$ is an isomorphism.

1.3. **Dimension three: modern results.** Problem 1.1 is harder when $\dim X = 3$. One must take $Y \in |L|$ to be *very general*, avoiding a countable union of Noether-Lefschetz components, which are dense in the Euclidean topology [7, 9]; moreover, the conclusion fails without additional positivity assumptions on L. We briefly survey the work done since 1980; see our survey [2] for more details.

1.3.1. *Carlson, Green, Griffiths and Harris, 1983:* Infinitesimal variant for sufficiently ample L on smooth n-fold X using infinitesimal variations of Hodge structures [8]. They prove that in the family of smooth $Y \in |L|$, the infinitesimally fixed part of the middle cohomology groups $H^{p,q}(Y)$ is precisely the fixed cohomology coming from the ambient space X.

1.3.2. *Green, 1984:* used Koszul cohomology [14, 15] to show that Noether-Lefschetz components satisfy $d-3 \leq \operatorname{codim}(V, |\mathcal{O}_{\mathbb{P}^3}(d)|) \leq p_g(d) = \binom{d-1}{3}$ [13]. In particular, all Noether-Lefschetz components for quartics have codimension one. His original argument used a spectral sequence to deduce a vanishing of a Koszul cohomology group, but in 1988 he gave a slicker proof using a filtration [16].

1.3.3. *Griffiths and Harris, 1985:* proved Theorem 1.3 by degenerating a general degree d surface to a union of a plane union a smooth surface of degree $d-1$ and computing the Picard group of the central fiber of a desingularization of the total family [17].

1.3.4. *Ein, 1985:* extended Noether's theorem from line bundles to vector bundles of higher rank [12]. If $T \subset H^0(E)$ is a t-dimensional subspace and E is a rank r bundle, one obtains a map $T \otimes \mathcal{O}_X \to E$ and dependency loci D_k where the rank of this map is at most k. If $2(r+3-t) > n$, then D_{t-2} is empty and $Y = D_{t-1}$ is smooth: assuming sufficient ampleness, Ein computed $\operatorname{Pic} Y$ in terms of $\operatorname{Pic} X$.

1.3.5. *Lopez, 1989:* For very general surfaces $S \subset \mathbb{P}^3$ containing a smooth curve C, $\operatorname{Pic} S$ is freely generated by C and $\mathcal{O}_S(1)$ [27].

1.3.6. *Ciliberto, Harris, Miranda and Green, 1988:* showed that the Noether-Lefschetz components are dense in the Euclidean topology [9].

1.3.7. *Ciliberto and Lopez, 1991:* constructed components of varying codimensions [10].

1.3.8. *Joshi, 1995:* used ideas in unpublished notes of Mohan Kumar and Srinivas to prove a new infinitesimal variant for smooth threefolds, obtaining a result for general *singular* surfaces [24].

1.3.9. *Ravindra and Srinivas, 2009:* proved that for X normal, $\operatorname{Cl} X \to \operatorname{Cl} Y$ is an isomorphism for very general Y if L ample and $K(L)$ globally generated [35].

1.3.10. *Brevik and Nollet, 2011:* proved a version for class groups and base locus (similar to Theorem 1.10) for $X = \mathbb{P}^3$ [1].

1.4. **Moishezon's theorem.** The best result for smooth complex threefolds was obtained by Moishezon in his general study of algebraic homology classes [28]. He adapted the argument of Lefschetz to prove the following remarkable theorem:

Theorem 1.11. *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth threefold and let $Y \subset X$ be a very general hyperplane section. Then the restriction* $\operatorname{Pic} X \to \operatorname{Pic} Y$ *is an isomorphism if and only if*

(a) $b_2(Y) = b_2(X)$ *or*
(b) $h^{2,0}(X) < h^{2,0}(Y)$.

When $X = \mathbb{P}^3$, condition (a) (equivalently $H^2(X, \mathbb{C}) \cong H^2(Y, \mathbb{C})$), picks up the "missing" case $L = \mathcal{O}(1)$ and $Y \subset \mathbb{P}^3$ is a plane. Since (a) fails for sufficiently positive L, the most important case is (b). The Hodge condition $h^{2,0}(X) < h^{2,0}(Y)$ means $h^2(\mathcal{O}_Y) > h^2(\mathcal{O}_X)$ or the Serre dual $h^0(K_Y) > h^1(K_X)$, but in view of the exact sequence

$$0 \to H^0(K_X) \to H^0(K_X \otimes L) \to H^0(K_Y) \to H^1(K_X) \to 0$$

arising from adjunction and Kodaira vanishing, this is equivalent to $h^0(K_X) < h^0(K_X \otimes L)$. Since L is very ample and $\dim X > 0$, this is equivalent to $H^0(K_X \otimes L) \neq 0$ by the following result for varieties X of positive dimension:

Fact 1.12. *If $A, L \in \operatorname{Pic} X$, L very ample, then $H^0(L \otimes A) \neq 0 \Rightarrow h^0(A) < h^0(L \otimes A)$.*

The hypothesis $H^0(K_X \otimes L) \neq 0$ is notably weaker than those of several theorems in the previous section. It is also essentially the hypothesis for the variant in Voisin's book on Hodge theory [41, 42]. Adapting the argument of Griffiths and Harris [17], we give a new proof of Theorem 1.11 (b) when $L = \mathcal{O}(1)$ is a product of very ample line bundles:

Theorem 1.13. *1 Let X be a smooth complex threefold. If $A, B \in \operatorname{Pic} X$ are very ample and $H^0(K_X \otimes A \otimes B) \neq 0$, then $r : \operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} Y$ for very general $Y \in |A \otimes B|$.*

Proof. Similar to the proof of Griffiths and Harris for $X = \mathbb{P}^3$, we focus on a general linear pencil $\mathbb{P}^1 \subset |A \otimes B|$ containing a smooth surface S and a reducible surface $T \cup P$ at $t = 0$ with $D = T \cap P$ a smooth curve. Our situation is more difficult because $\operatorname{Pic} X \to \operatorname{Pic} P$ need not be an isomorphism and $\operatorname{Pic}^0 X$ need not be zero. The total family $M \subset X \times \mathbb{P}^1$ is singular over the central degenerate fiber $0 = t \in \mathbb{P}^1$ at the points of intersection $S \cap P \cap T$, but the corresponding total family of strict transforms $\tilde{M} \subset \tilde{X} \times \mathbb{P}^1$ where $\tilde{X} \to X$ is the blow-up at $S \cap T$ is nonsingular near $t = 0$. The central fiber becomes $T \cup \tilde{P}$, where $\tilde{P} \to P$ is the blowup along $P \cap T \cap S$.

Claim 1: $\operatorname{Pic} \tilde{M}_0 \cong \operatorname{Pic} X \oplus \mathbb{Z}\mathcal{O}_{\tilde{M}}(\tilde{P})|_{\tilde{M}_0}$.

Combining the hypothesis $H^0(K_X \otimes L) \neq 0$ and Fact 1.12 with the exact sequence

$$0 \to H^0(K_X \otimes B) \to H^0(K_X \otimes B \otimes A) \to H^0(K_P \otimes B) \to 0$$

yields $H^0(K_P \otimes B) \neq 0$. Combining this with Fact 1.12 and the exact sequence

$$0 \to H^0(K_P) \to H^0(K_P \otimes B) \to H^0(K_D) \to H^1(K_P) \to 0$$

yields $H^1(\mathcal{O}_D) = H^0(K_D) > H^1(K_P) = H^1(\mathcal{O}_P)$. Since $H^1(\mathcal{O}_V)$ is naturally the tangent space to the Picard variety $\operatorname{Pic}^0 V$ at the origin, the inclusion leads us to a closed immersion $\operatorname{Pic}^0 P \hookrightarrow \operatorname{Pic}^0 D$ of Picard varieties. In particular the embedding of P by B is not the Veronese surface embedded by quadrics nor is it a ruled surface, hence $|B \otimes \mathcal{O}_P|$ has a pencil consisting of irreducible curves [27, II.2.4], from which it follows that if $L \in \operatorname{Pic} P$ and $L_D \cong \mathcal{O}_D$ for general D in such a pencil, then $L \cong \mathcal{O}_P$ is trivial [27, II.2.3]. Looking at the countably many representatives $L \in \operatorname{Pic} P / \operatorname{Pic}^0 P - \{0\}$ in the Néron-Severi group, it follows that $L_D \not\cong D$ for a proper algebraic subset of $D \in |B|$ and hence $\operatorname{Pic} P \to \operatorname{Pic} D$ is injective for very general $D \in |B \otimes \mathcal{O}_P|$. Reversing the roles of P and T we also obtain injective $\operatorname{Pic} T \to \operatorname{Pic} D$ for very general $D \in |A \otimes \mathcal{O}_T|$ and for very general $(P, T) \in |A| \times |B|$ we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} X & \longrightarrow & \operatorname{Pic} P \\ \downarrow & & \downarrow \\ \operatorname{Pic} T & \longrightarrow & \operatorname{Pic} D \end{array} \tag{2}$$

in which all restriction maps are injective (the maps from $\operatorname{Pic} X$ are injective by [34]).

We show Diagram (2) is Cartesian. For this we fix P and $L \in \operatorname{Pic} P/\operatorname{Pic}^0 P$ and show that if $L|_D \in \operatorname{Pic} T$ for general $T \in |B|$, then $L \in \operatorname{Pic} X$. The idea of the proof here is that the set of T with $L|_D \in \operatorname{Pic} T$ is closed and if is all of $|B|$, then by unicity we can find a continuous family of lines bundles $L_t \in \operatorname{Pic} T_t$. Restricting to a pencil $\mathbb{P}^1 \subset |B|$, representability of the relative Picard scheme gives a line bundle on the total family over the pencil, but if the pencil has base locus $C = T_0 \cap T_1$, then the total family is isomorphic to $\tilde{X} \to X$, the blow up along C. This gives a line bundle A on X modulo the exceptional divisor of the blow up, and with minor modification we show that $A|_P = L$. Working over the countable representatives for $\operatorname{Pic} P/\operatorname{Pic}^0 P$ we find that the diagram is Cartesian for very general T and hence $\operatorname{Pic} P \times_{\operatorname{Pic} D} \operatorname{Pic} T \cong \operatorname{Pic} X$, which computes $\operatorname{Pic} M_0$. With similar arguments we show for very general S that $\operatorname{Pic} \tilde{M}_0 \cong \operatorname{Pic} X \oplus \mathcal{O}_{\tilde{M}}(\tilde{P}) \otimes \mathcal{O}_{\tilde{M}_0}$.

With Claim 1 in hand, the rest of the proof flows along the lines of Griffiths and Harris' proof [17]. Each Noether-Lefschetz component $V \subset |A \otimes B|$ is the image of a relative Hilbert scheme component $W \subset \mathbf{Hilb}$ of curves whose divisor classes are not in the image of $\operatorname{Pic} X$ and we need to show that the map $\pi : W \to |A \otimes B|$ is not dominant. If it were dominant, then there is a pencil $\mathbb{P}^1 \subset |A \otimes B|$ as constructed in Step 1 for which $\pi^{-1}\mathbb{P}^1 \to \mathbb{P}^1$ is dominant. We can then find an integral curve $E_0 \subset W$ dominating $\mathbb{P}^1$ and after normalizing a smooth curve $E \to \mathbb{P}^1$. Pulling back the family $\tilde{M} \to \mathbb{P}^1$ back to a new family $Z \to E$, we use Claim 1 to deduce that image is proper. □

2. Questions of Srinivas

The local ring $A = \mathcal{O}_{X,x}$ of a point x on a complex algebraic variety X is a *geometric local domain*. If A is normal, then so is the completion $R = \hat{A}$ and the natural map $A \to R$ is flat, hence the *Mori map* [39]

$$\iota : \operatorname{Cl} A \hookrightarrow \operatorname{Cl} R \tag{3}$$

given by $p \mapsto \sum_{P \cap A = p} e(P,p)P$ is a well-defined injective homomorphism where $e(P,p)$ is the ramification index of the field extension $K(A/p) \subset K(B/P)$. It is essentially the pull-back map along $\operatorname{Spec} R \to \operatorname{Spec} A$ after removing singularities. Srinivas [38] asks about the images of Inclusion (3) for fixed R as A varies over geometric normal local domains satisfying $R \cong \hat{A}$:

Question 2.1. Let R be the completion of a normal geometric local ring. Which subgroups of $\operatorname{Cl} R$ arise as images $\operatorname{Cl} A \hookrightarrow \operatorname{Cl} R$ where $R \cong \widehat{A}$?

The following example of Srinivas [38] shows that Question 2.1 is interesting.

Example 2.2. The complete local ring $R = \mathbb{C}[[x,y,z]]/(x^2 + y^3 + z^7)$ has class group $\operatorname{Cl} R \cong \mathbb{C}$ [39], but for every geometric local ring A with $\widehat{A} \cong R$, the image $\operatorname{Cl} A \hookrightarrow \operatorname{Cl} R$ is necessarily finitely generated [38, Example 3.9]. Srinivas reasons that if $A = \mathcal{O}_{X,x}$ for a surface X and $Y \to X$ is a resolution of singularities, then the induced map $\operatorname{Pic} Y \to \operatorname{Cl} A$ is surjective. Since $\operatorname{Pic}^0 Y$ is projective, it has trivial image in the affine group $\mathbb{G}_a = \mathbb{C}$, therefore $\operatorname{Cl} A \to \mathbb{C}$ factors through the finitely generated Neron-Severi group.

After asking Question 2.1, Srinivas backs off somewhat, instead asking about the minimal possible images of the Mori map.

Question 2.3. Let R be the completion of a normal geometric local ring.

(a) If R is Gorenstein, is R the completion of a geometric UFD?
(b) Does there exist geometric ring A with $R \cong \hat{A}$ and $\operatorname{Cl} A = \langle \omega_A \rangle$?

Remark 2.4. These questions are only interesting for singularities. If A is a regular local ring, then so is $R = \hat{A}$ so that $\operatorname{Cl} R = 0$ and consequently any ring A with $\hat{A} \cong R$ satisfies $\operatorname{Cl} A = 0$, hence is a UFD.

Remark 2.5. If A is the codimension r quotient of a regular local ring B, then the dualizing module $\omega_A = \operatorname{Ext}^r_A(B, A) \in \operatorname{Cl} R$ is independent of B and the image in $\operatorname{Cl} R$ is independent of A. Thus Question 2.1 (b) asks about the minimal image. Moreover $\omega_A \in \operatorname{Cl} R$ is zero if and only if R is Gorenstein [30], so (a) is a special case of (b).

2.1. **Minimal images.** We note some progress made on Question 2.3.

2.1.1. *Grothendieck, 1968:* solved Samuel's conjecture, proving that a local complete intersection ring that is factorial in codimension ≤ 3 is a UFD [18, XI, Cor. 3.14].

2.1.2. *Hartshorne and Ogus, 1973:* if R has an isolated singularity, $\operatorname{depth} R \geq 3$ and embedding dimension at most $2 \dim R - 3$, then R is a UFD [22].

2.1.3. *Heitmann, 1993:* characterized completions of UFDs [23], but his constructions rarely produce geometric rings.

2.1.4. *Srinivas, 1987:* Part (a): Yes for rational double points [37].

2.1.5. *Parameswaran and van Straten, 1993:* Part (b): Yes if $\dim R = 2$ [33].

2.1.6. *Parameswaran and Srinivas, 1994:* Part (a): Yes for local complete intersections of dimensions two and three with isolated singularity [32].

2.1.7. *Brevik and Nollet, 2016:* Part (a): Yes for hypersurface singularities of dimension ≥ 2 [3] and local complete intersection singularities of dimension ≥ 3 [4].

2.2. **Proof for hypersurfaces.** We illustrate our method for hypersurface singularities. Let $f \in \mathbb{C}[x_1, x_2, \dots x_n]$ be the equation of a hypersurface V which normal at the origin p, corresponding to the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, and let $R = \mathbb{C}[[x_1, x_2, \dots x_n]]/(f)$ be the completion of $A = \mathcal{O}_{V,p}$. The singular locus D of V is given by the ideal $(f) + J_f$, where $J_f = (f_{x_1}, \dots, f_{x_n})$. Primary decomposition in $\mathbb{C}[x_1, \dots, x_n]$ gives

$$J_f = \bigcap_{p_i \subset \mathfrak{m}} q_i \cap \bigcap_{p_i \not\subset \mathfrak{m}} q_i$$

where q_i is p_i-primary and we have sorted into components that meet the origin and those that do not. Denote by K the intersection on the left and J the intersection on the right; localizing at $\mathfrak{m}$ we find that $(J_f)_{\mathfrak{m}} = K_{\mathfrak{m}}$ because $J_{\mathfrak{m}} = (1)$.

Now if $K = (k_1, \dots, k_r) \subset \mathbb{C}[x_1, \dots, x_n]$, then the closed subscheme Z defined by the ideal $I_Y = (f, k_1^3, \dots k_r^3)$ is supported on the components of the singular locus of V that contain the origin, hence $\operatorname{codim}(Z, \mathbb{P}^n) \geq 3$ by normality of V at the origin. The very general hypersurface Y containing Z satisfies $\operatorname{Cl} Y = 0$ by Theorem 1.10 or 1.3.10 above, so $\operatorname{Cl} \mathcal{O}_{Y,p} = 0$ as well and $\mathcal{O}_{Y,p}$ is a UFD [20, Prop. 6.2]. Moreover, Y has local equation

$$g = f + a_1 k_1^3 + \cdots + a_r k_r^3$$

for units a_i, and hence $f - g \in K^3$. Since $K_{\mathfrak{m}} = (J_f)_{\mathfrak{m}}$, their completions are equal in $\mathbb{C}[[x_1, \dots, x_n]]$. Therefore $f - g \in J_f^3 \subset \mathfrak{m} J_f^2$ and a result of Ruiz [36, V, Lemma 2.2] tells us that $\widehat{\mathcal{O}}_{Y,p} = \mathbb{C}[[x_1, \dots, x_n]]/(g) \cong \mathbb{C}[[x_1, \dots, x_n]]/(f) = R$.

2.3. **General images.** While Question 2.3 has received much attention, Question 2.1 remains wide open. To understand it better, we call an element $\alpha \in \operatorname{Cl} R$ a *geometric divisor* if it is in the image of the inclusion (3) for some geometric local ring A with $R = \widehat{A}$. In view of Example 2.2, we pose the following:

Question 2.6. Which statements hold for the completion R of a normal geometric ring?

(a) Given any finitely generated group $G \subset \operatorname{Cl} R$, there a geometric local ring B with $\widehat{B} = R$ and $G = \operatorname{Cl} B$.
(b) Given $\alpha_1, \dots, \alpha_r \in \operatorname{Cl} R$, there is a geometric local ring B with $\widehat{B} = R$ and $\alpha_i \in \operatorname{Cl} B$ for $1 \le i \le r$.
(c) Every $\alpha \in \operatorname{Cl} R$ a geometric divisor.
(d) The geometric divisors form a subgroup of $\operatorname{Cl} R$.

Note the easy implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. Our ignorance about the nature of geometric divisors is revealed in part (c): could there be transcendental divisors that cannot be accessed geometrically? The methods of [3, 4] suggest the following possibility:

Conjecture 2.7. *Statement 2.6 (a) holds for local complete intersection singularities.*

For local complete intersection singularities we recently [6] proved the reverse implication $(b) \Rightarrow (a)$:

Theorem 2.8. *If $x \in X \subset \mathbb{P}^n$ is a normal complete intersection point and $G \subset \operatorname{Cl} \mathcal{O}_{X,x}$ is finitely generated, then there exists a complete intersection $W \subset \mathbb{P}^n$ and $w \in W$ with $\hat{\mathcal{O}}_{W,w} \cong \hat{\mathcal{O}}_{X,x}$ and $\operatorname{Cl} \mathcal{O}_{W,w} \subset \operatorname{Cl} \mathcal{O}_{W,w}$ identified with $G \subset \operatorname{Cl} \hat{\mathcal{O}}_{X,x}$.*

With current Noether-Lefschetz theorems [1, 4] we must take $n = 3$ when $\dim X = 2$ in the theorem statement, but we expect the statement holds in general.

Corollary 2.9. *Statement 2.6 (a) holds for rational double point singularities.*

We proved this result first using explicit non-reduced curves as base loci in our Noether-Lefschetz theorem with base locus [3], but it follows more easily from Theorem 2.8.

Remark 2.10. Corollary 2.9 contrasts with the case in which the function field is rational, where Mohan Kumar [29] shows that for most $\mathbf{A}_n$ and $\mathbf{E}_n$ singularities there is only one isomorphism class for the local ring (and thus the class group). The three exceptions, with two possibilities each, are the $\mathbf{E}_8$, $\mathbf{A}_7$, and $\mathbf{A}_8$; for all other $\mathbf{E}_n$ and $\mathbf{A}_n$ singularities, the Mori map is an isomorphism. Since an $\mathbf{E}_8$ is a UFD under completion, any $\mathbf{E}_8$ is a UFD. By following Mohan Kumar's constructions of the $\mathbf{A}_7$ and $\mathbf{A}_8$ carefully, one sees that the image of the Mori map for the $\mathbf{A}_7$ is either the full completed class group $\mathbb{Z}/8\mathbb{Z}$

or the subgroup of order 4, while in the $\mathbf{A}_8$ case the Mori map is either surjective onto $\mathbb{Z}/9\mathbb{Z}$ or its image is of order 3.

We deduce the following for vertex singularities on cones over smooth varieties.

Corollary 2.11. *Statement 2.6 (a) holds for the completed local ring at the vertex p of the cone V over smooth complete intersection varieties $X \subset \mathbb{P}^n$ of dimension at least three.*

Finally we show that every finitely generated abelian group arises as a local class group of a singularity of a surface in $\mathbb{P}^3$.

Corollary 2.12. *Let G be any finitely generated abelian group. Then there is a point p on a normal surface $S \subset \mathbb{P}^3$ for which $G \cong \operatorname{Cl} \mathcal{O}_{S,p} \subset \operatorname{Cl} \widehat{\mathcal{O}_{S,p}}$.*

We construct S with an isolated singularity p that is analytically isomorphic to a vertex singularity of a cone over a plane curve of high degree. Write $G \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^s \mathbb{Z}/n_i\mathbb{Z}$ for suitable r, s, n_i. Choose a smooth plane curve C of high degree with genus satisfying $g \geq \frac{1}{2}(r+s)$. The vertex p of the cone S over C has class group $\operatorname{Cl} \mathcal{O}_{S,p} \cong \operatorname{Pic} C/\langle \mathcal{O}_C(1)\rangle$. Since the only degree-0 class in $\langle \mathcal{O}_C(1)\rangle$ is 0, the composite map

$$\operatorname{Pic}^0(C) \to \operatorname{Pic} C \to \operatorname{Pic} C/\langle \mathcal{O}_C(1)\rangle$$

is injective, where $\operatorname{Pic}^0(C)$ is the subgroup of $\operatorname{Pic} C$ consisting of the degree-0 classes. Since $\operatorname{Pic}^0(C)$ is isomorphic to the Jacobian variety $J(C)$, which for the complex curve C is isomorphic to $\mathbb{C}^g/\Lambda$ with Λ a rank-$(2g)$ lattice in $\mathbb{C}^g$, we see that

$$\operatorname{Pic}^0(C) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong (\mathbb{R}/\mathbb{Z})^{2g}$$

as an additive group. Since $\mathbb{R}/\mathbb{Z}$ has elements of all orders (including ∞), we can choose r elements of summands having order ∞ and s elements having respective orders n_i, which generate a subgroup of $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ isomorphic to G. Apply Theorem 2.8 to these elements.

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