PERTURBATIONS OF EQUIVARIANT DIRAC OPERATORS

IGOR PROKHORENKOV AND KEN RICHARDSON

Abstract. Let M be a compact Riemannian manifold with an action by isometries of a compact Lie group G. Suppose that this action could be lifted to an action by isometries on a Clifford bundle E over M. We use the method of the Witten deformation to compute the virtual representation-valued index of a transversally elliptic Dirac operator on E. We express the multiplicities of the associated representation in terms of the local action of G near the singular set of the deformation. A complete answer is obtained when $G = S^1$.

1. The Dirac Operator

Here we give an idea of what the standard Dirac operator is.

Recall that the Laplacian in $\mathbb{R}^3$ is the second order differential operator

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}.$$

This is an extremely useful operator in physics. Examples include the heat equation

$$\frac{\partial u}{\partial t} + \Delta u = 0$$

and the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

Dirac wanted to find a square root for the Laplacian, for reasons of physics. Well, let's see. It should be a 1st order differential operator. Oh, let's just try to guess what it should be: call it D (for Dirac). Let's try

$$D = \sum_{j=1}^{3} c_j \partial_j = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} + c_3 \frac{\partial}{\partial x_3}$$

with a, b, c constants – for starters. We compute:

$$\begin{aligned} D^2 &= \left(\sum_{j=1}^{3} c_j \partial_j \right)^2 \\ &= c_1^2 \frac{\partial^2}{\partial x_1^2} + c_2^2 \frac{\partial^2}{\partial x_2^2} + c_3^2 \frac{\partial^2}{\partial x_3^2} + (c_1 c_2 + c_2 c_1) \frac{\partial^2}{\partial x_1 \partial x_2} + (c_1 c_3 + c_2 c_3) \frac{\partial^2}{\partial x_2 \partial x_3} + ... \\ &= \Delta ??? \end{aligned}$$

Date: October, 2003.

1991 *Mathematics Subject Classification.* 58J20; 58J37; 58J50.

Key words and phrases. Equivariant index, group action, Witten deformation, perturbation, singularity, Dirac operator, localization.

So, it looks like it is not likely to work. Hey, unless you use quaternions. Anyway, here is the answer:

$$c_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; c_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; c_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

These are called the Pauli spin matrices or Clifford matrices. The resulting algebra is called the Clifford algebra. The resulting Dirac operator is

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_2} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{\partial}{\partial x_3}$$

Dirac operators are very nice. They are self-adjoint, elliptic operators, and they have discrete spectrum on suitably chosen domains. What does elliptic mean? It means that the principal symbol

$$\begin{aligned} \sigma(D)(y) &= \sum_{j=1}^{3} c_j (i y_j) \\ &= i y_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + i y_2 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + i y_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} -y_3 & -i y_1 - y_2 \\ i y_1 - y_2 & y_3 \end{pmatrix} \end{aligned}$$

is invertible for every nonzero vector y. To see this, note that

$$\det \begin{pmatrix} -y_3 & -i y_1 - y_2 \\ i y_1 - y_2 & y_3 \end{pmatrix} = -y_1^2 - y_2^2 - y_3^2.$$

In layman's terms, elliptic operators are those operators that differentiate in all possible directions. Elliptic operators are important for many reasons — they have finite-dimensional kernels and eigenspaces, and those spaces consist of smooth sections (vector-valued functions).

In general, the Dirac operator on a Hermitian vector bundle E over an n-dimensional Riemannian manifold M is

$$D = \sum_{j=1}^{n} c_j \nabla_{e_j}^{E},$$

where each $c_j = c(e_j)$ is a section of $\mathrm{End}(E)$ ($c : TM \to \mathrm{End}(E)$), and $\{e_j\}$ is a local orthonormal frame in TM. The c_j satisfy the same relations; that is

$$\begin{aligned} c(v)\, c(w) + c(w)\, c(v) &= -2 \langle v, w \rangle, \text{ also} \\ \nabla_w^E (c(V)\, s) &= c\left(\nabla_w^M V\right) s + c(V) \nabla_w^E s \end{aligned}$$

for any $v, w \in TM$, $V \in \Gamma(TM)$, $s \in \Gamma(E)$. In the case $M = \mathbb{R}^3$ or any quotient thereof, we use the matrices above to define Clifford multiplication in general as

$$c(v) = c(v_1, v_2, v_3) = \begin{pmatrix} i v_3 & -v_1 + i v_2 \\ v_1 + i v_2 & -i v_3 \end{pmatrix}.$$

Examples of a Dirac operator include D above on sections of $\mathbb{C}^2 \to T^3$, or on $\mathbb{C}^2 \to S^2$:

$$\begin{aligned} D^{S^2} &= c\left(\frac{\partial}{\partial \phi}\right)\frac{\partial}{\partial \phi} + \frac{1}{\sin\phi} c\left(\frac{1}{\sin\phi}\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta} \\ &= c\left(v_\phi\right) v_\phi + \frac{1}{\sin^2\phi} c\left(v_\theta\right) v_\theta \\ &= \begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix} \frac{\partial}{\partial\phi} + \frac{1}{\sin\phi}\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\frac{\partial}{\partial\theta_{xy}} \end{aligned}$$

Check:

$$\begin{aligned} \begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}^2 &= \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix} &= \begin{pmatrix} i\cos\phi & -(\sin\phi)\, e^{-i\theta} \\ (\sin\phi)\, e^{i\theta} & -i\cos\phi \end{pmatrix} \\ \begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix} &= \begin{pmatrix} -i\cos\phi & (\sin\phi)\, e^{-i\theta} \\ -(\sin\phi)\, e^{i\theta} & i\cos\phi \end{pmatrix} \end{aligned}$$

Also

$$\Delta^{S^2} = -\frac{\partial^2}{\partial\phi^2} - \cot\phi\frac{\partial}{\partial\phi} - \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.$$

In a sense, the Dirac operators are the most fundamental among first-order elliptic operators. Given any first-order, elliptic differential operator on $\mathbb{R}^n$, there is a Dirac operator that is stably homotopic to it. The word “homotopic” refers to a homotopy (i.e. continuous family) of elliptic differential operators, and the word “stably” refers to the bundle enlargement process or its reverse: given an elliptic D on $\mathbb{C}^n$-valued functions (i.e. - sections), the operator $D \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^n \otimes \mathbb{C}^2 \cong \mathbb{C}^{2n}$ is also elliptic. For this reason the Dirac operators are important in index theory. The index of an elliptic operator $L^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ on sections of a graded Hermitian vector bundle $E = E^+ \oplus E^-$ over a closed Riemannian manifold M is defined to be

$$\operatorname{ind}\left(L^+\right) = \ker\left(L^+\right) - \ker\left(L^-\right),$$

where L^- is defined to be the $L^2(M, E)$-adjoint of L^+. By the celebrated Atiyah-Singer Index Theorem, this index can be expressed in terms of geometric and topological data; the simplest example of this yields the Gauss-Bonnet Theorem of differential geometry:

$$\operatorname{ind}\left(d + \delta|_{\Omega^{\mathrm{even}}(M)\to\Omega^{\mathrm{odd}}(M)}\right) = \chi(M) = \frac{1}{2\pi}\int_M K,$$

where $d+\delta$ is the sum of the exterior derivative and its adjoint, $\chi(M)$ is the Euler characteristic of the closed surface M, K is the Gauss curvature. The index of an elliptic operator is invariant under stable homotopy, so we can always reduce such problems to Dirac operator index problems (at least locally). There are other ways to see that the index of any elliptic operator on any manifold can be reduced to a calculation of the index of a Dirac operator (on a possibly different manifold).

2. Examples of transversally elliptic operators

In this section we describe constructions of transversally elliptic differential operators. The setting for a transversally elliptic operator is as follows. Suppose that the compact Lie group G acts on the closed, Riemannian manifold M by isometries; suppose that in addition we have a Hermitian vector bundle E over M on which G acts unitarily (in a way compatible with the action on M). Such an action induces an action on $\Gamma(M,E)$ by $(gs)(x) := g(s(g^{-1}x))$. A differential operator D is called *equivariant* if it commutes with the action of G; in particular such operators preserve the space $\Gamma(M,E)^G$ of G-invariant sections. Let $(T_GM)_x$ denote the subspace of the tangent space T_xM that is the normal space to the orbit Gx. Note that the dimension of this space can vary with $x \in M$. We say that an equivariant operator D is *transversally elliptic* if the symbol $\sigma(D)$ is invertible on all nonzero covectors in each $((T_GM)_x)^*$. In layman's terms, we only require that D act like an elliptic operator in directions orthogonal to the orbits.

A simple example of a transversally elliptic operator is as follows. Consider the torus $T^3 = \mathbb{Z}^3 \diagdown \mathbb{R}^3$, the unit cube with opposite faces identified. Next, let $G = S^1$ act by $\theta(x_1,x_2,x_3) = \left(x_1,x_2,x_3+\frac{\theta}{2\pi}\right) \text{mod} 1$. Then the orbits are the circles where x_1 and x_2 are fixed and x_3 varies. Then the truncated Dirac operator

$$D^{\mathrm{tr}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_2}$$

is transversally elliptic. This "truncation" is actually a general procedure that works in special cases of group actions, which we now describe.π

2.1. **Foliation case.**

Suppose that the compact Lie group G acts on the closed, Riemannian manifold M by isometries, such that the isotropy subgroups all have the same dimension. Equivalently, the orbits have the same dimension. In this case, the orbits form a Riemannian foliation $\mathcal{F}$ of M such that the metric is bundlelike, meaning that the leaves (orbits) are locally equidistant. In such foliations, there is a natural construction of transversal Dirac operators (see Bruning-Kamber-Richardson, Glazebrook-Kamber, Lazarov), described as follows. Choose a local adapted frame field $\{e_1,...,e_n\}$ for the tangent bundle of M, such that $\{e_1,...,e_q\}$ is a local basis of the normal bundle $N\mathcal{F}$ for the foliation and such that each e_j is a basic vector field for $1 \leq j \leq q$. The word *basic* means that the flows of those vector fields map leaves to leaves, and such a basis can be chosen near every point if and only if the foliation is Riemannian. Next, assume that we have a complex Hermitian vector bundle $E \to M$ that is a bundle of $\mathbb{C}l(N\mathcal{F})$ modules that is equivariant with respect to the G action, and let ∇ be the corresponding equivariant, metric, Clifford connection. We define the transversal Dirac operator D^{tr} by

$$D^{\mathrm{tr}} := \sum_{j=1}^{q} c(e_j)\nabla_{e_j}.$$

This definition is independent of the choices made; in fact it is the composition of the maps

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\pi_{\mathcal{F}}} \Gamma(N^*\mathcal{F} \otimes E) \xrightarrow{\cong} \Gamma(N\mathcal{F} \otimes E) \xrightarrow{c} \Gamma(E),$$

where $\pi_{\mathcal{F}}$ is the map induced by the projection $T^*M \to N^*\mathcal{F}$, and where c denotes Clifford multiplication. This operator is transversally elliptic and maps the space of invariant sections to itself. However, it is generally not self adjoint on the space of invariant sections. The simple modification

$$\widetilde{D} = D^{\mathrm{tr}} - \frac{1}{2} c(H)$$

is a self-adjoint operator, where H is the mean curvature vector field of the orbits.

2.2. **General case - Lifted Transversal Dirac Operator.** Suppose that the compact Lie group G acts on the closed, connected Riemannian manifold M in a more general way, where the dimensions of the orbits are not all the same. We assume that G acts by isometries.

Given a G-manifold, we first may the group action to an action on the bundle $O_M \xrightarrow{p} M$ of orthonormal frames over M. The induced action of $g \in G$ on TM is the differential of the diffeomorphism, and this action induces an action on O_M. Observe that the action of G on O_M is regular, meaning that the isotropy subgroups corresponding to any two points of M are conjugate. This can be seen as follows. Let H be the isotropy subgroup of a frame $f \in O_M$. Then H also fixes $p(f) \in M$, and since H fixes the frame, its differentials fix the entire tangent space at $p(f)$. Since it fixes the tangent space, every element of H also fixes every frame in $p^{-1}(p(f))$; thus every frame in a given fiber must have the same isotropy subgroup. Since the elements of H map geodesics to geodesics and preserve distance, a neighborhood of $p(f)$ is fixed by H. Thus, H is a subgroup of the isotropy subgroup at each point of that neighborhood. Conversely, if an element of G fixes a neighborhood of a point x in M, then it fixes all frames in $p^{-1}(x)$, and thus all frames in the fibers above that neighborhood. Since M is connected, we may conclude that every point of O_M has the same isotropy subgroup H, and H is the subgroup of G that fixes every point of M. Since this subgroup is normal, we often reduce the group G to the group G/H so that our action is effective, in which case the isotropy subgroups on O_M are all trivial.

In any case, the G orbits on O_M are diffeomorphic and form a Riemannian fiber bundle, in the natural metric on O_M defined as follows. We require that fibers are orthogonal to the horizontal subspaces coming from the Levi-Civita connection, the metric on the fibers comes from the normalized, biinvariant metric on $O(n)$, and the horizontal metric is the pullback of the metric on M. The Riemannian submersion from the frame bundle O_M to the orbit space $W = O_M \diagup G$ induces a natural metric on W.

We will use the structure described above to define natural transversal Dirac operators on the general G-manifold M. Suppose that we have a Hermitian vector bundle $E \to O_M$ that is equivariant with respect to both the G action and the $O(n)$ action; since both groups are subgroups of the isometry group of O_M, there are many geometrically and topologically defined bundles with this property. Let N_G denote the normal bundle to the foliation of G-orbits on O_M, and suppose that in addition E is a bundle of $\mathbb{C}l(N_G)$ modules. We construct the transversal Dirac operator D^{tr} as in the last section. Let $\mathcal{E}$ denote the vector bundle over M defined as follows. Given $x \in M$, define

$$\mathcal{E}_x := \Gamma\left(E|_{p^{-1}(x)} \right)^{O(n)},$$

the space of $O(n)$-invariant sections of E restricted to the fiber above x. This forms a finite dimensional vector space whose dimension is less than or equal to the rank of E. We denote by $\mathcal{E}$ the vector bundle over M whose fibers are $\mathcal{E}_x$; a maximal linearly independent set of $O(n)$-invariant sections of E defined near $p^{-1}(x)$ give the local trivializations of $\mathcal{E}_x$. There is a natural invertible map $\Phi : \Gamma(\mathcal{E}) \to \Gamma(E)^{O(n)}$ defined by $\Phi(s)(w) = s(p(w))|_w$. Observe that Φ is an L^2 isometry, and thus its adjoint is its inverse. Next, we define the operator $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ by the formula

$$D = \Phi^{-1} \circ D^{\mathrm{tr}} \circ \Phi.$$

Note that this operator is well defined, since D^{tr} maps $\Gamma(E)^{O(n)}$ to itself. This operator is transversally elliptic.

2.3. **General Case - Reduced Dirac Operator.** Another naturally defined transversal Dirac operator is defined as follows. Again, suppose that the compact Lie group G acts on the closed, connected Riemannian manifold M , where the dimensions of the orbits are not all the same. We assume that G acts by isometries. Let $\{V_1, ..., V_k\}$ be a set of vector fields on M induced from an orthonormal basis of $\mathfrak{g}$. Let E be a graded, self-adjoint $\mathbb{C}\mathrm{l}(TM)$ module over M, and let $c : TM \to \mathrm{Hom}(E, E)$ denote the Clifford multiplication. Let $D : \Gamma(M, E) \to \Gamma(M, E)$ be ordinary Dirac operator associated to this data. We define the *reduced Dirac operator* $\widehat{D}$ by

$$\widehat{D} := D - \sum_{j=1}^{k} c(V_j) \nabla_{V_j}.$$

It is clear that this definition is independent of the choice of orthonormal basis of $\mathfrak{g}$. Also, it is clear that this operator is transversally elliptic, but it fails to be elliptic in general.

2.4. **Locally Transverse Dirac operators.** We again assume that the compact Lie group G acts on the closed, connected Riemannian manifold M , where the dimensions of the orbits are not all the same, and we assume that G acts by isometries. We assume G is endowed with the normalized, biinvariant metric. Suppose that $L : \Gamma(M, E) \to \Gamma(M, E)$ is a differential operator on sections of a G-equivariant vector bundle E over M. We say that L is a *locally transverse Dirac operator* if near any point $p \in M$, it has the following form. Suppose that the orbit pG of p has codimension q in M. Let H denote the isotropy subgroup of p, and let $H^{\perp}$ denote the subspace of $\mathfrak{g}$ consisting of vectors orthogonal to the submanifold $H \subset G$ at the identity. Then the subgroup $\exp\left(H^{\perp}\right)$ of G acts on M, and its orbits near p form a Riemannian foliation of codimension q, and the connected component of pG coincides with $p\exp\left(H^{\perp}\right)$. Then we require that the action of L on invariant sections of E coincide with a locally defined transversal Dirac operator corresponding to the Riemannian foliation defined near p. That is, given a local adapted orthonormal basis $\{e_1, ..., e_q\}$ for the normal bundle to the orbits of $\exp\left(H^{\perp}\right)$ near p that is locally invariant, then

$$Lu = \left(\sum_{j=1}^{q} c(e_j) \nabla_{e_j}\right) u,$$

for any section u of E defined near p such that $\mathcal{L}_X u = 0$ for every $X \in H^{\perp}$. Here, $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

3. Examples

Example 3.1. *Let S^1 act on S^2 by rotations. Let E be the standard spin^c bundle over $S^2 = \mathbb{C}P^1$.*

First we will construct a lifted transversal Dirac operator on sections of E. Let $p : O_{S^2} \to S^2$ be the oriented orthonormal frame bundle. Observe that O_{S^2} is isometric to $SO(3)$, which *is isometric to* $\mathbb{R}P^3$ *in appropriate invariant metrics (see [14]). In particular, the metric at any point of* $SO(3) \subset M_3(\mathbb{R})$ *is*

$$\langle A, B\rangle = \operatorname{tr}\left(A^t B\right).$$

To see this isometry, observe that a point of O_{S^2} *is a unit vector in* $\mathbb{R}^3$ *and a unit vector orthogonal to it; a third vector may be added canonically to produce an oriented orthonormal frame in* $\mathbb{R}^3$*. We will consider matrices of* $SO(3)$ *to be elements of* O_{S^2} *by using the first row as the position vector and the following rows as the framing of the tangent space. Observe that the lifted action corresponds to the action on* $SO(3)$ *given by multiplication on the right by*

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Vectors on $SO(3)$ *may be expressed as elements of the Lie algebra*

$$\mathfrak{o}(3) = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\},$$

and the tangent space to the S^1 *action is the span of the left-invariant vector field induced by* $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *at the identity. Thus, the normal bundle of the corresponding foliation on* $SO(3)$ *is trivial and is the subbundle* N_{S^1} *of* $T(SO(3))$ *that is given by the subspace* $N_{S^1}|_A$ *of* $M_3(\mathbb{R})$ *at the matrix* $A \in SO(3)$*, where*

$$N_{S^1}|_A = \left\{ A \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{pmatrix} \middle| \, b, c \in \mathbb{R} \right\} \subset M_3(\mathbb{R}).$$

Note that the vectors $V_1 = A \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ *and* $V_2 = A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ *are orthogonal to the orbit direction* $T = A \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *in the* $SO(3)$ *metric:*

$$\begin{aligned} \langle V_1, T\rangle &= tr\left(\left(A \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right)^T A \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= tr\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^T A^T A \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= tr\left(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0. \end{aligned}$$

Similarly,

$$\langle V_2, T\rangle = \langle V_1, V_2\rangle = 0.$$

Next, we consider the trivial bundle $\mathbb{C}^2 \to SO(3)$ *and the action of* $\mathrm{Cl}(\mathbb{R}^2) \cong \mathrm{Cl}(N_{S^1})$ *on* $\mathbb{C}^2$ *with* $c(V_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $c(V_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. *Using this data and the directional derivatives , we produce the transversal Dirac operator*

$$D^{\mathrm{tr}} := \sum_{j=1}^{2} c(V_j)\,\partial_{V_j},$$

where ∂_{V_j} *denotes the directional derivative in direction* V_j. *Now, as in Section 2.2, we consider for* $x \in S^2$ *the set*

$$\begin{aligned} \mathcal{E}_x &= \left.\Gamma\left(SO(3), \mathbb{C}^2\right)^{SO(2)}\right|_{p^{-1}(x)} \\ &= \Gamma\left(p^{-1}(x), \mathbb{C}^2\right)^{SO(2)} \\ &\cong \mathbb{C}^2 \end{aligned}$$

Thus, $\mathcal{E}$ *is the trivial bundle* $\mathbb{C}^2 \to S^2$. *Letting* $A = (A_{ij}) \in SO(3)$, *note that* $p_*(V_1)_A$ *is the vector on tangent space to the sphere at* $p(A) = (A_{11}, A_{12}, A_{13})$ *that satisfies*

$$\begin{aligned} p_*(V_1)_A &= p_*\left(A\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right) \\ &= p_*\begin{pmatrix} -A_{13} & 0 & A_{11} \\ -A_{23} & 0 & A_{21} \\ -A_{33} & 0 & A_{31} \end{pmatrix} \\ &= (-A_{13}, 0, A_{11}) \\ &= \left.\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\right|_{(A_{11}, A_{12}, A_{13})} \end{aligned}$$

Similarly,

$$p_*(V_2)_A = \left.\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\right|_{(A_{11}, A_{12}, A_{13})}$$

Thus, the lifted transversal Dirac operator on sections of $\mathbb{C}^2 \to S^2$ is

$$\begin{aligned}
D &= \Phi^{-1} \circ D^{\mathrm{tr}} \circ \Phi = \sum_{j=1}^{2} c\left(V_j\right) \partial_{p_* V_j} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right) \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta_{xz}} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial \theta_{yz}} \\
&= \begin{pmatrix} 0 & e^{-i\theta_{xy}} \frac{\partial}{\partial \phi} + ie^{i\theta_{xy}} \cot\phi \frac{\partial}{\partial \theta_{xy}} \\ -e^{i\theta_{xy}} \frac{\partial}{\partial \phi} + ie^{-i\theta_{xy}} \cot\phi \frac{\partial}{\partial \theta_{xy}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{-i\theta_{xy}} \\ -e^{i\theta_{xy}} & 0 \end{pmatrix} \frac{\partial}{\partial \phi} + \cot\phi \begin{pmatrix} 0 & -ie^{-i\theta_{xy}} \\ -ie^{i\theta_{xy}} & 0 \end{pmatrix} \frac{\partial}{\partial \theta_{xy}},
\end{aligned}$$

where is the θ_{xz} is the polar angular coordinate in the xz-plane, θ_{yz} is the polar angular coordinate in the yz-plane, and ϕ is the distance from the north pole. Note that this operator fails to be elliptic precisely at the equator $z = 0$, because the vector fields $\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)$ and $\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)$ are collinear exactly at points where $z = 0$.

Observe that this operator is not self-adjoint, although its principal symbol is (because each $\frac{\partial}{\partial \theta_\cdot}$ is skew-adjoint and each matrix is skew adjoint).

Note that the kernel of this operator restricted to the space of smooth sections of $\mathbb{C}^2$ is infinite dimensional. Suppose we try to solve the equation

$$Du = 0.$$

Suppose $u = \begin{pmatrix} u_1(x,y,z) \\ u_2(x,y,z) \end{pmatrix} \in \ker D$, then observe that

$$\begin{aligned}
D\begin{pmatrix} u_1(x,y,z) \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \left(-z\frac{\partial u_1}{\partial x} + x\frac{\partial u_1}{\partial z}\right) + i\left(-z\frac{\partial u_1}{\partial y} + y\frac{\partial u_1}{\partial z}\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \left(-z\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right) u_1\left(x, y, \sqrt{1-x^2-y^2}\right) \end{pmatrix},
\end{aligned}$$

in the upper hemisphere. Thus, in the upper hemisphere, u_1 is holomorphic as a function of the coordinates (x,y). Similarly, u_2 is antiholomorphic with respect to the holomorphic coordinates $z = x+iy$ in the upper hemisphere. The same facts are true for the restriction of u to the lower hemisphere. Note that any such function that is defined on the unit disk and continuous on the closure is determined by its values on the boundary. Thus, the function u_1 and u_2 are symmetric with respect to the map $z \to -z$. We conclude that the smooth sections $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ in $\ker D$ are exactly functions of x and y alone such that u_1 is holomorphic and u_2 is antiholomorphic.

Example 3.2. *Let S^1 act on S^2 by rotations, as above. We will now examine the reduced Dirac operator $\widehat{D}$ on S^2 with the given S^1 action, as in Section 2.3. First, observe that the standard spin^c bundle over S^2 is just the trivial bundle $\mathbb{C}^2 \to S^2$, where the action of*

$\mathbb{Cl}(TS^2)$ on $\mathbb{C}^2$ is given by the restriction of the action of $\mathbb{Cl}(\mathbb{R}^3)$ on $\mathbb{C}^2$. That is, the action of the vector $v = (v_1, v_2, v_3) \in T_xS^2 \subset \mathbb{R}^3$ is given by multiplication by the matrix

$$C(v) = v_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + v_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus, the full Dirac operator on S^2 is

$$D = C(e_1)\,\partial_{e_1} + C(e_2)\,\partial_{e_2},$$

where $\{e_1, e_2\}$ is a local orthonormal basis for the tangent space and ∂_{e_i} denotes the directional derivative in direction e_i. Next, let $\frac{\partial}{\partial\theta_{xy}}$ denote the induced fundamental vector field on S^2 that comes from the S^1 action, so that

$$\frac{\partial}{\partial\theta_{xy}} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

By the definition in Section 2.3, the reduced Dirac operator is

$$\begin{aligned} \widehat{D} &= D - c\left(\frac{\partial}{\partial\theta_{xy}}\right)\nabla_{\frac{\partial}{\partial\theta_{xy}}} \\ &= D - c\left(\frac{\partial}{\partial\theta_{xy}}\right)\frac{\partial}{\partial\theta_{xy}}. \end{aligned}$$

Locally, at any point $(x, y, z) \in S^2 \diagdown \{z = \pm 1\}$, for convenience we consider the orthonormal frame $\left\{\frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta_{xy}}\right\}$, where ϕ is the angular coordinate (measured from the z-axis). In this basis,

$$D = C\left(\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\phi} + C\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta_{xy}}\right)\frac{1}{\sin\phi}\frac{\partial}{\partial\theta_{xy}},$$

so

$$\begin{aligned} \widehat{D} &= C\left(\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\phi} + C\left(\frac{1}{\sin\phi}\frac{\partial}{\partial\theta_{xy}}\right)\frac{1}{\sin\phi}\frac{\partial}{\partial\theta_{xy}} - C\left(\frac{\partial}{\partial\theta_{xy}}\right)\frac{\partial}{\partial\theta_{xy}} \\ &= C\left(\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\phi} + \left(\cot^2\phi\right)C\left(\frac{\partial}{\partial\theta_{xy}}\right)\frac{\partial}{\partial\theta_{xy}} \\ &= \left[\cos\theta_{xy}\cos\phi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sin\theta_{xy}\cos\phi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \sin\phi\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right]\frac{\partial}{\partial\phi} \\ &\quad + \left(\cot^2\phi\right)\left[-\sin\theta_{xy}\sin\phi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \cos\theta_{xy}\sin\phi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right]\frac{\partial}{\partial\theta_{xy}} \\ &= \begin{pmatrix} -i\sin\phi & -e^{-i\theta_{xy}}\cos\phi \\ e^{i\theta_{xy}}\cos\phi & i\sin\phi \end{pmatrix}\frac{\partial}{\partial\phi} + \cos\phi\cot\phi\begin{pmatrix} 0 & ie^{-i\theta_{xy}} \\ ie^{i\theta_{xy}} & 0 \end{pmatrix}\frac{\partial}{\partial\theta_{xy}} \end{aligned}$$

This operator fails to be elliptic exactly at the points where $\phi = \frac{\pi}{2}$, at the equator. Note that the equation

$$\left(\begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}\frac{\partial}{\partial\phi} + \cos\phi\cot\phi\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\frac{\partial}{\partial\theta}\right)\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = 0$$

implies
$\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix}$ *is a constant vector. Also,*

$$\left(\begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}\frac{\partial}{\partial\phi} + \cos\phi\cot\phi\begin{pmatrix} 0 & ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}\frac{\partial}{\partial\theta}\right)\begin{pmatrix} e^{in\theta}f(\phi) \\ e^{in\theta}g(\phi) \end{pmatrix} = 0$$

implies

$$\left(\begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}\frac{\partial}{\partial\phi} - n\cos\phi\cot\phi\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}\right)\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = 0,$$

which implies

$$\left(\frac{\partial}{\partial\phi} - n\cos\phi\cot\phi\begin{pmatrix} -i\sin\phi & -e^{-i\theta}\cos\phi \\ e^{i\theta}\cos\phi & i\sin\phi \end{pmatrix}^{-1}\begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}\right)\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = 0, \text{ or}$$

$$\left(\frac{\partial}{\partial\phi} - n\cos\phi\cot\phi\begin{pmatrix} \cos\phi & i(\sin\phi)e^{-i\theta} \\ -i(\sin\phi)e^{i\theta} & -\cos\phi \end{pmatrix}\right)\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = 0$$

for all values of θ*, so in particular if we differentiate the equation with respect to* θ *at* $\theta = 0$*, we get*

$$-n\cos\phi\cot\phi\begin{pmatrix} 0 & \sin\phi \\ \sin\phi & 0 \end{pmatrix}\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = 0, \text{ so}$$

$\begin{pmatrix} f(\phi) \\ g(\phi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *Thus, there are no solutions of the form* $\begin{pmatrix} e^{in\theta}f(\phi) \\ e^{in\theta}g(\phi) \end{pmatrix}$ *for* $n \neq 0$. *Note that* $\frac{\partial}{\partial\theta}$ *commutes with* $\widehat{D}$*, so we would expect that the kernel of* $\widehat{D}$ *would decompose into eigenspaces of* $\frac{\partial}{\partial\theta}$*. These calculations have shown that the only sections* u *that are solutions to the equation*

$$\widehat{D}u = 0$$

are the constant sections of $\mathbb{C}^2 \to S^2$*. Note the contrast with the previous example of the lifted transversal Dirac operator above, which has an infinite-dimensional kernel.*

Example 3.3. *Let* S^1 *act on* S^2 *by rotations, as above. We now construct examples of locally transverse Dirac operators as in Section 2.4. One such example is the reduced Dirac operator* $\widehat{D} = D - c\left(\frac{\partial}{\partial\theta_{xy}}\right)\nabla_{\frac{\partial}{\partial\theta_{xy}}}$ *above. Another similarly constructed example would be*

$$L = D - M(\theta_{xy}, \phi)\nabla_{\frac{\partial}{\partial\theta_{xy}}},$$

where $M(\theta_{xy}, \phi)$ *is any smooth section of* $M_2(\mathbb{C}) \to S^2$ *that maps* $S^{\pm}$ *to* $S^{\mp}$*, where* $S^{\pm}$ *are the subbundles of* $\mathbb{C}^2$ *corresponding to the graded spinor bundle. Since* $M(\theta_{xy}, \phi)\nabla_{\frac{\partial}{\partial\theta_{xy}}}$ *is zero on sections invariant by the group action, that term is irrelevant to the "locally transverse" condition.*

4. Transversally elliptic operators and their index

Let us now discuss the general problem of localizing a multiplicity of the index of a transversally elliptic operator. First of all, let's set the stage: $D^{\pm} : \Gamma(M, E^{\pm}) \to \Gamma(M, E^{\mp})$ is a first order differential operator with $D^{-} = (D^{+})^{*}$ that is equivariant and transversally elliptic with respect to the action of a connected, compact group G of isometries of $(M, E^{\pm})$. The group G acts on $\Gamma(M, E^{\pm})$ by $(sg)(x) = s(xg^{-1})g$. Let $\rho : G \to \mathrm{End}(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho : G \to \mathbb{C}$ be its character, $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. By the Peter-Weyl Theorem, the functions $\{\chi_\rho\}_\rho$ are eigenfunctions of the Laplacian on G

and form an orthonormal set in $L^2(G)$ with the normalized, biinvariant metric. The multiplicity $\operatorname{ind}^{\rho}(D^+) \in \mathbb{Z}$ is defined to be the multiplicity of the representation ρ in the (infinite dimensional) representation of G on $\ker D^+$ minus the multiplicity of ρ in the representation of G on $\ker D^-$. If ρ_0 is the trivial representation of G, then we have

$$\begin{aligned}\operatorname{ind}^{\rho_0}\left(D^+\right) &:= \dim\left(\left(\ker D^+\right)^G\right) - \dim\left(\left(\ker D^-\right)^G\right)\\ &= \operatorname{ind}^G\left(D^+\right) := \operatorname{index}\left(\left.D^+\right|_{\Gamma(M,E^+)^G\to\Gamma(M,E^-)^G}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{ind}^{\rho}\left(D^+\right) &:= \text{multiplicity of } \rho \text{ in } \ker D^+ - \text{multiplicity of } \rho \text{ in } \ker D^-\\ &= \frac{1}{\dim V_\rho}\operatorname{index}\left(\left.D^+\right|_{\Gamma(M,E^+)^\rho\to\Gamma(M,E^-)^\rho}\right),\end{aligned}$$

where $\Gamma(M,E^\pm)^\rho$ is the space of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M,E^\pm)$ that are unitarily equivalent to the ρ representation. The above equalities are valid because the action of G on $\Gamma(M,E^\pm)$ commutes with $D^\pm$. We note that these multiplicities are finite for the reasons given in the next paragraph. This fact is well-known and was first shown by Atiyah in [1].

Let $\{X_1,...,X_r\}$ be an orthonormal basis of the Lie algebra of G. Let $\mathcal{L}_{X_j}$ denote the induced Lie derivative with respect to X_j on sections of E, and let $C = \sum_j \mathcal{L}_{X_j}^*\mathcal{L}_{X_j}$ be the Casimir operator on sections of E. Letting exp denote the exponential map on G, $(\cdot,\cdot)$ be the pointwise Hermitian inner product on E, and letting $s, s' \in \Gamma(M,E^\pm)$, observe that

$$\begin{aligned}\left\langle \mathcal{L}_{X_j}s, s'\right\rangle &= \left.\frac{\partial}{\partial t}\right|_{t=0}\int_M \left(s\left(y\exp\left(-tX_j\right)\right)\exp\left(tX_j\right), s'(y)\right)\operatorname{dvol}(y)\\ &= \left.\frac{\partial}{\partial t}\right|_{t=0}\int_M \left(s\left(y\exp\left(-tX_j\right)\right), s'(y)\exp\left(-tX_j\right)\right)\operatorname{dvol}(y)\\ (\text{with } u = y\exp(-tX_j)) &= \left.\frac{\partial}{\partial t}\right|_{t=0}\int_M \left(s(u), s'\left(u\exp\left(tX_j\right)\right)\exp\left(-tX_j\right)\right)\operatorname{dvol}(u)\\ &= \left\langle s, \mathcal{L}_{-X_j}s'\right\rangle = \left\langle s, -\mathcal{L}_{X_j}s'\right\rangle,\end{aligned}$$

so that $C = -\sum_j \left(\mathcal{L}_{X_j}\right)^2$. Furthermore, since the Laplacian on $C^\infty(G)$ is given by $\Delta = -\sum_j X_j^2$, we have the following formulas relating C and Δ. Given two sections $s, s' \in \Gamma(M,E^\pm)$, let $f : G \to \mathbb{C}$ be the function defined by $f(g) = \langle sg, s'\rangle$. Then

$$\begin{aligned}\left\langle Cs, s'\right\rangle &= \Delta f(\mathbf{1});\\ \Delta f(g) &= \left\langle C(sg), s'\right\rangle.\end{aligned}$$

Since C is induced from the action of G, C commutes with $D^\pm$ and thus acts on the eigenspaces of $D^\pm$. Likewise, G acts on the eigenspaces of C. Note that $Cs = 0$ if and only if $s \in \Gamma(M,E)^G$, so that

$$\begin{aligned}\left(\ker D^\pm\right)^G &= \left(\ker\left(D^\mp D^\pm\right)\right)^G\\ &= \left(\ker\left(D^\mp D^\pm + tC\right)\right)^G,\end{aligned}$$

which is finite dimensional, because $D^{\mp}D^{\pm}+tC$ is elliptic for sufficiently large t. Thus, $\mathrm{ind}^{\rho_0}(D^+)$ is a well-defined integer. More generally, suppose that V_ρ is an irreducible component of $\Gamma(M,E^{\pm})^{\rho}$ and that the set of sections $\{s_k\}$ is an orthonormal basis of V_ρ. Since G acts on the eigenspaces of C, we may assume that V_ρ is a subspace of the eigenspace of C corresponding to an eigenvalue μ of the Casimir operator C. For every $g\in G$, the character satisfies $\chi_\rho(g)=\sum_k \langle s_k g, s_k\rangle$ and $\Delta\chi_\rho=\lambda_\rho\chi_\rho$, so that

$$\begin{aligned}\lambda_\rho\chi_\rho(\mathbf{1}) &= \lambda_\rho\sum_k\langle s_k,s_k\rangle\\ &= \sum_k\langle C(s_k),s_k\rangle=\mu\sum_k\langle s_k,s_k\rangle .\end{aligned}$$

Therefore, $\mu=\lambda_\rho$, and C acts on $\Gamma(M,E^{\pm})^{\rho}$ by multiplication by λ_ρ. The argument also shows that every eigenspace of C corresponding to eigenvalue μ is a direct sum of copies of the representation ρ, where $\lambda_\rho=\mu$. In other words, $\Gamma(M,E^{\pm})^{\rho}$ is precisely the λ_ρ-eigenspace of C. Thus,

$$\begin{aligned}\ker\left(D^{\pm}\big|_{\Gamma(M,E^{\pm})^{\rho}}\right) &= \ker\left(D^{\mp}D^{\pm}\big|_{\Gamma(M,E^{\pm})^{\rho}}\right)\\ &= \ker\left(D^{\mp}D^{\pm}+t\left(C-\lambda_\rho\mathbf{1}\right)\big|_{\Gamma(M,E^{\pm})^{\rho}}\right),\end{aligned}$$

which is finite dimensional, because $D^{\mp}D^{\pm}+t(C-\lambda_\rho\mathbf{1})$ is elliptic on $\Gamma(M,E^{\pm})$ for sufficiently large t. This argument shows that the multiplicities $\mathrm{ind}^{\rho}(D^+)$ are well-defined integers. Also, by standard asymptotic arguments, the multiplicities cannot grow faster than polynomially as a function of the eigenvalue λ_ρ.

The relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\mathrm{ind}_g(D^+)$ is as follows. The virtual character $\mathrm{ind}_g(D^+)$ is given by (see [1])

$$\begin{aligned}\mathrm{ind}_g\left(D^+\right) &:= \text{“}\mathrm{tr}\left(g|_{\ker D^+}\right)-\mathrm{tr}\left(g|_{\ker D^-}\right)\text{”}\\ &= \sum_\rho \mathrm{ind}^{\rho}\left(D^+\right)\chi_\rho(g)\in D(G),\end{aligned}$$

where $D(G)$ is the set of distributions on G. Note that the trace above does not make sense as a function, since $\ker D^{\pm}$ is in general infinite-dimensional, but it does make sense as a distribution on G . Similarly, the sum above does not in general converge, but it makes sense as a distribution on G. That is, if dg is the normalized, biinvariant volume form on G, and if $\phi=\sum c_\rho\chi_\rho\in C^\infty(G)$, then

$$\begin{aligned}\mathrm{ind}_*\left(D^+\right)(\phi) &= \text{“}\int_G \mathrm{ind}_g\left(D^+\right)\overline{\phi(g)}\,dg\text{”}\\ &= \sum_\rho \mathrm{ind}^{\rho}\left(D^+\right)\int\chi_\rho(g)\overline{\phi(g)}\,dg\\ &= \sum_\rho \mathrm{ind}^{\rho}\left(D^+\right)\overline{c_\rho},\end{aligned}$$

an expression which converges because c_ρ is rapidly decreasing and $\mathrm{ind}^{\rho}(D^+)$ is grows at most polynomially as ρ varies over the irreducible representations of G. From this calculation, we see that the multiplicities determine Atiyah's distributional index. Conversely, let $\alpha: G\to$

End (V_α) be an irreducible unitary representation. Then

$$\begin{aligned}\operatorname{ind}_*\left(D^+\right)(\chi_\alpha) &= \sum_\rho \operatorname{ind}^\rho\left(D^+\right)\int \chi_\rho(g)\,\overline{\chi_\alpha(g)}\,dg \\ &= \operatorname{ind}^\alpha D^+,\end{aligned}$$

so that complete knowledge of the equivariant distributional index is equivalent to knowing all of the multiplicities $\operatorname{ind}^\rho(D^+)$. Note that Atiyah showed in [1] that the operator $D^+|_{\Gamma(M,E^+)^\rho\to\Gamma(M,E^-)^\rho}$ is Fredholm, and thus its index $\operatorname{ind}^\rho(D^+)$ depends only on the homotopy class of the principal transverse symbol of D^+.

Let us now consider the heat kernel expression for the index multiplicities. Given a representation $\rho : G \to V_\rho$, we wish to compute

$$\begin{aligned}\operatorname{ind}^\rho\left(D^+\right) &= \frac{1}{\dim V_\rho}\operatorname{index}\left(D^+|_{\Gamma(M,E^+)^\rho\to\Gamma(M,E^-)^\rho}\right) \\ &= \frac{1}{\dim V_\rho}\left(\ker\left(D^-D^+|_{\Gamma(M,E^+)^\rho}\right) - \ker\left(D^+D^-|_{\Gamma(M,E^-)^\rho}\right)\right) \\ &= \frac{1}{\dim V_\rho}\begin{pmatrix}\dim\left(\ker\left(D^-D^+ + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\cap\Gamma\left(M,E^+\right)^\rho\right) \\ -\dim\left(\ker\left(D^+D^- + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\cap\Gamma\left(M,E^-\right)^\rho\right)\end{pmatrix}.\end{aligned}$$

Note that if E_μ denotes the eigenspace corresponding to eigenvalue μ, the map

$$D^+ : E_\mu\left(D^-D^+ + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\cap\Gamma\left(M,E^+\right)^\rho \to E_\mu\left(D^+D^- + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\cap\Gamma\left(M,E^+\right)^\rho$$

is an isomorphism as long as $\mu \neq 0$. Thus, the usual McKean-Singer argument shows that for every $t > 0$ and sufficiently large $u > 0$,

$$\begin{aligned}\left(\dim V_\rho\right)\operatorname{ind}^\rho\left(D^+\right) &= \operatorname{tr}\left(e^{-t\left(D^-D^+ + u\left(C-\lambda_\rho\mathbf{1}\right)\right)}\Big|_{\Gamma(M,E^+)^\rho}\right) - \operatorname{tr}\left(e^{-t\left(D^+D^- + u\left(C-\lambda_\rho\mathbf{1}\right)\right)}\Big|_{\Gamma(M,E^-)^\rho}\right) \\ &= e^{u\lambda_\rho t}\left(\operatorname{tr}\left(e^{-t\left(D^-D^+ + uC\right)}\Big|_{\Gamma(M,E^+)^\rho}\right) - \operatorname{tr}\left(e^{-t\left(D^+D^- + uC\right)}\Big|_{\Gamma(M,E^-)^\rho}\right)\right).\end{aligned}$$

More generally, if $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a smooth, rapidly decreasing function such that $\psi(0) = 1$, then

$$\begin{aligned}\left(\dim V_\rho\right)\operatorname{ind}^\rho\left(D^+\right) &= \operatorname{tr}\left(\psi\left(D^-D^+ + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\big|_{\Gamma(M,E^+)^\rho}\right) - \\ &\quad\operatorname{tr}\left(\psi\left(D^+D^- + u\left(C-\lambda_\rho\mathbf{1}\right)\right)\big|_{\Gamma(M,E^-)^\rho}\right).\end{aligned}$$

Now, observe the general fact about operators A that commute with a G action. If $K(x,y)$ is the kernel of the operator A, then

$$(Af)(x) = \int_M K(x,y)\,f(y)\;\mathrm{dvol}(y).$$

Next, observe that the operation $f(x) \longmapsto (Pf)(x) := \int_G f(xg)\,dg$ commutes with A , so that

$$\begin{aligned}\int_M K(x,y)\,(Pf)(y)\;\mathrm{dvol}(y) &= \int_M P_x K(x,y)\,f(y)\;\mathrm{dvol}(y) \\ &= \int_M\left(\int_G K(xg,y)\;dg\right)f(y)\;\mathrm{dvol}(y).\end{aligned}$$

If f is an invariant section, then $Pf = f$, so that

$$(Af)(x) = \int_M \left(\int_G K(xg, y)\ dg \right) f(y)\ \mathrm{dvol}(y).$$

Hence, $\left(\int_G K(xg, y)\ dg\right)$ is a kernel for A^G, the operator A restricted to invariant sections. Thus,

$$\mathrm{tr}\left(A^G\right) = \int_{M\times G} K(xg, x)\ dg\, \mathrm{dvol}(x).$$

Similarly, if α is an irreducible representation of G with character χ_α, the operation $f(x) \longmapsto (P^\alpha f)(x) := \int_G f(xg)\ \chi_\alpha(g)\ dg$ commutes with A, so that

$$\begin{aligned} \int_M K(x,y)\,(P^\alpha f)(y)\ \mathrm{dvol}(y) &= \int_M P^\alpha_x K(x,y)\, f(y)\ \mathrm{dvol}(y) \\ &= \int_M \left(\int_G K(xg, y)\ \chi_\alpha(g)\ dg \right) f(y)\ \mathrm{dvol}(y). \end{aligned}$$

If f is a section in $\Gamma(M, E)^\alpha$, then $P^\alpha f = f$, so that

$$(Af)(x) = \int_M \left(\int_G K(xg, y)\ \chi_\alpha(g)\ dg \right) f(y)\ \mathrm{dvol}(y).$$

Hence, $\left(\int_G K(xg, y)\ \chi_\alpha(g)\ dg\right)$ is a kernel for A^α, the operator A restricted to $\Gamma(M, E)^\alpha$. Thus,

$$\mathrm{tr}\left(A^\alpha\right) = \int_{M\times G} K(xg, x)\ \chi_\alpha(g)\ \ dg\, \mathrm{dvol}(x).$$

By applying this formula to the traces above, we have

$$\begin{aligned} (\dim V_\rho)\, \mathrm{ind}^\rho\left(D^+\right) &= \int_{M\times G} e^{u\lambda_\rho t}\ \left(K^+(t, xg, x) - K^-(t, xg, x)\right) \chi_\alpha(g)\ \ dg\, \mathrm{dvol}(x) \\ &= \int_{M\times G} \left(K^{\psi,+}_\rho(xg, x) - K^{\psi,-}_\rho(xg, x)\right) \chi_\alpha(g)\ \ dg\, \mathrm{dvol}(x) \\ &\quad \mathrm{tr}\left(\psi\left(D^-D^+ + u\left(C - \lambda_\rho \mathbf{1}\right)\right)\big|_{\Gamma(M,E^+)^\rho}\right) - \\ &\quad \mathrm{tr}\left(\psi\left(D^+D^- + u\left(C - \lambda_\rho \mathbf{1}\right)\right)\big|_{\Gamma(M,E^-)^\rho}\right) \end{aligned}$$

where $K^\pm(t, x, y)$ is the kernel for $e^{-t\left(D^\mp D^\pm + uC\right)}$ on $\Gamma(M, E^\pm)$, and $K^{\psi,\pm}_\rho(x, y)$ is the kernel for $\psi\left(D^-D^+ + u\left(C - \lambda_\rho\mathbf{1}\right)\right)$ on $\Gamma(M, E^\pm)$.

5. Localization Theorem

Now we consider a more general situation, when $(D_s)^2 - D^2 = (D + sZ)^2 - D^2 = s(ZD + DZ) + s^2Z^2$ is a first order operator. In general, the kernel of D_s does not necessarily localize at the critical points of the zeroth order operator Z (the points where the rank of Z drops) as $s \to \infty$. For example, let V be an equivariant vector field, and let $D_s = D + sic(V)$; note that $Z = ic(V)$ is a self-adjoint zeroth order operator. If V has isolated, nondegenerate zeros, then $ic(V)$ has isolated critical points. Note that $Z^2 = |V|^2$. In this Section we prove the localization theorem for D_s.

In the following, let $\|\cdot\|_{k,2}$ denote the Sobolev $(k, 2)$ norm.

Lemma 5.1. *(Transversally Elliptic Estimates) Let D be a first order, strongly transversally elliptic operator that maps the space $\Gamma(M,E)^{\rho}$ to itself, and suppose that the restriction of D to $\Gamma(M,E)^{\rho}$ is formally self-adjoint. Then there exist positive constants c_1^{ρ}, c_2^{ρ} such that for every $\omega \in \Gamma(M,E)^{\rho}$,*

(1) $\left\|(D^2+1)\,\omega\right\|_{k,2} \geq c_2^{\rho}\left\|\omega\right\|_{k+2,2} \geq c_1^{\rho}\left\|\omega\right\|_{k+1,2}$
(2) *If the coefficients of the operator D depend on a real parameter s and are bounded by a polynomial in s as $s \to \infty$, then the corresponding constants c_1^{ρ} and c_2^{ρ} are bounded by a polynomial in s as $s \to \infty$.*

Proof. Since the restriction of D to $\Gamma(M,E)^{\rho}$ is formally self-adjoint, $\left\|(D^2+1)\,\omega\right\|_{0,2}^2 = \langle D^2\omega, D^2\omega\rangle + 2\langle D\omega, D\omega\rangle + \langle\omega,\omega\rangle$, and thus we have

$$\left\|\left(D^2+1\right)\omega\right\|_{0,2} \geq \left\|\omega\right\|_{0,2}. \tag{5.1}$$

Let $\{X_1, ..., X_r\}$ be an orthonormal basis of the Lie algebra of G. Let $\mathcal{L}_{X_j}$ denote the induced Lie derivative with respect to X_j on sections of E, and let $C = \sum_j \mathcal{L}_{X_j}^{*}\mathcal{L}_{X_j}$ be the Casimir operator on sections of E. Let λ_{ρ} be the eigenvalue of C corresponding to the eigenspace $\Gamma(M,E)^{\rho}$. For any $\omega \in \Gamma(M,E)^{\rho}$, $(D^2+1)\,\omega = (D^2 + u\,(C-\lambda_{\rho}\mathbf{1}) + 1)\,\omega$ for any $u>0$, and for sufficiently large $u>0$, $(D^2 + u\,(C-\lambda_{\rho}\mathbf{1}) + 1)$ is elliptic on $\Gamma(M,E)$. Thus, the ordinary elliptic estimates imply that

$$\left\|\left(D^2+1\right)\omega\right\|_{k,2} \geq m_2^k\left\|\omega\right\|_{k+2,2} - m_0^k\left\|\omega\right\|_{k,2}. \tag{5.2}$$

for some constants $m_2^k > 0$ and $m_0^k \geq 0$ for each $k \geq 0$, independent of ω. Using (5.1) and (5.2) with $k=0$, the proposed inequality holds for $k=0$. If the coefficients of the operator D depend on a real parameter s and are bounded by a polynomial in s, then it is clear from the construction that the constants in the inequalities above are bounded by polynomials in s. Thus, the conclusion of the lemma is true for $k=0$.

Assume the conclusion is true for $0 \leq k \leq r$. Then

$$\begin{aligned}\left\|\left(D^2+1\right)\omega\right\|_{r+1,2} &\geq a_1\left\|\left(D^2+1\right)^2\omega\right\|_{r-1,2} \text{ by the definition of Sobolev norm}\\ &\geq a_2\left\|\left(D^2+1\right)\omega\right\|_{r,2} \text{ by the induction hypothesis}\\ &\geq a_3\left\|\omega\right\|_{r+2,2} \text{ by the induction hypothesis,}\end{aligned} \tag{5.3}$$

for positive constants a_1, a_2, and a_3. The ordinary elliptic estimates for $(D^2 + u\,(C-\lambda_{\rho}\mathbf{1}) + 1)$ imply that

$$\left\|\left(D^2+1\right)\omega\right\|_{r+1,2} \geq c_3\left\|\omega\right\|_{r+3,2} - c_2\left\|\omega\right\|_{r+2,2}$$

for constants $c_3 > 0$ and $c_2 \geq 0$. By (5.3),

$$\left\|\left(D^2+1\right)\omega\right\|_{r+1,2} \geq \left(\frac{c_3}{1+\frac{c_2}{a_3}}\right)\left\|\omega\right\|_{r+3,2}.$$

The conclusion concerning the dependence on s is again clear. The result follows by induction on k. □

With the notation as above, let $k_s(x,y)$ denote the kernel of the operator $\phi(D_s)$ on $\Gamma(M,E)^{\rho}$, where ϕ is a positive, even Schwartz function such that $\phi(0)=1$ and the Fourier transform $\widehat{\phi}\,(\xi)$ is supported in the interval $[-\varepsilon,\varepsilon]$. Let $\mathrm{Crit}\,(V)=\{x\in M\,|\,V\,(x)=0\}$denote the set of zeros of V.

Proposition 5.2. *On the complement of a* 2ε*-neighborhood of* Crit(V)*, the kernel* $k_s\,(x,y)$ *of* $\phi\,(D_s)$ *restricted to* $\Gamma\,(M,E)^\rho$ *satisfies* $k_s\,(x,y)\to 0$ *uniformly as* $s\to\infty$*. That is,* $k_s\,(x,y)\to 0$ *uniformly for* y *in the complement of a* 2ε*-neighborhood of* Crit(V) *and* $x\in M$.

Proof. This proof is the same as [42] with some modifications. Choose a constant C so that $\|V\,(x)\|\geq C>0$ for all x in the complement of a ε-neighborhood of Crit(V). Then for every section $\beta\in\Gamma\,(M,E)^\rho$ that is supported on such a neighborhood and for sufficiently large s,

$$\left\langle D_s^2\beta,\beta\right\rangle\geq\frac{C^2s^2}{2}\left\langle\beta,\beta\right\rangle,D \tag{5.4}$$

by formula (**??**). Let $\mathcal{H}$ denote the L^2 closure of the set of sections in $\Gamma\,(M,E)^\rho$ supported on a ε-neighborhood of Crit(V). Then D_s^2 is a positive, symmetric operator on a dense subset of $\mathcal{H}$, so it extends to a self-adjoint operator A on $\mathcal{H}$ satisfying the same inequality above.

Let $\omega\in\Gamma\,(M,E)^\rho$ be supported on the complement of a 2ε -neighborhood of Crit$\,(V)$, and let

$$\omega_t=\cos\left(tD_s\right)\omega=\frac{1}{2}\left(e^{itD_s}+e^{-itD_s}\right)\omega,$$

which is a solution to the generalized wave equation $\left(\frac{\partial^2}{\partial t^2}+D_s^2\right)\omega_t=0$ corresponding to the operator D_s^2 with initial conditions $\omega_0=\omega$, $\frac{\partial}{\partial t}\omega_0=0$. The family of sections ω_t is the unique solution to this generalized wave equation (see [26], [36], and [46]). Note that the formula above implies that ω_t is an element of $\Gamma\,(M,E)^\rho$.

By the finite propagation speed property of symmetric hyperbolic systems (see [23]), the propagation speed of solutions to this equation is bounded from above by the supremum of the operator norms of the principal symbol of D_s, which is 1. Therefore, ω_t is identically zero on the ε-neighborhood of Crit$\,(V)$ if $|t|<\varepsilon$. This implies that $D_s^2\omega_t=A\omega_t$ for $|t|<\varepsilon$, so that ω_t is the unique solution to the system

$$\frac{\partial^2}{\partial t^2}\omega_t+A\omega_t=0;\quad\omega_0=\omega,\frac{\partial}{\partial t}\omega_0=0.$$

Since A is self-adjoint, we may use the functional calculus to write $\omega_t=\cos\left(t\sqrt{A}\right)\omega$.

For each nonnegative integer m, define ϕ_m by the formula

$$\phi_m(\lambda)=(1+\lambda^2)^{2m}\phi(\lambda);$$

note that each ϕ_m satisfies the same properties as ϕ; that is, ϕ_m is a positive, even Schwartz function such that $\phi_m\,(0)=1$ and the Fourier transform $\widehat{\phi_m}\,(\xi)$ is supported in the interval $[-\varepsilon,\varepsilon]$.

For a section $\omega \in \Gamma(M,E)^\rho$ that is supported on the complement of the 2ε -neighborhood of Crit(V) ,

$$\begin{aligned}\phi_m(D_s)\,\omega &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\hat{\phi}_m(t)\left(e^{itD_s}\omega\right)\,dt \\ &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\hat{\phi}_m(t)\left(\cos(tD_s)\,\omega\right)\,dt \text{ since } \hat{\phi}_m \text{ is even} \\ &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\hat{\phi}_m(t)\left(\cos\left(t\sqrt{A}\right)\omega\right)\,dt \\ &= \phi_m\left(\sqrt{A}\right)\omega. \end{aligned} \tag{5.5}$$

By (5.4), the operator $\sqrt{A}$ is positive and has operator norm is bounded below by $\frac{Cs}{\sqrt{2}}$ for s sufficiently large. Thus, the operator norm of $\phi_m\left(\sqrt{A}\right)$ (as an operator from $\mathcal{H}$ to itself) is bounded above by

$$c_m(s) = \sup\left\{|\phi_m(\lambda)| \,:\, \lambda \geq \frac{Cs}{\sqrt{2}}\right\}.$$

It is clear that $c_m(s)$ is rapidly decreasing as $s\to\infty$. By (5.5),

$$\|\phi_m(D_s)\,\omega\|_2 \leq c_m(s)\,\|\omega\|_2$$

for every $\omega \in \Gamma(M,E)^\rho$ supported on the complement of a 2ε -neighborhood of Crit(V).

Next, let $L^p = L^p(\Gamma(M,E)^\rho)$ denote the L^p-norm closure of $\Gamma(M,E)^\rho$, and let $W^k = W^k(\Gamma(M,E)^\rho)$ denote the closure of the space of such sections under the Sobolev $(k,2)$ -norm. By the transversally elliptic estimates (Lemma 5.1), the operator norm of $(1+D_s^2)^{-1}$: $W^k \longrightarrow W^{k+2}$ is bounded by a polynomial in s. The ordinary Sobolev imbedding theorem implies that $(1+D_s^2)^{-k} : L^2 \longrightarrow L^\infty$ is a bounded map whose operator norm is bounded by a polynomial in s if $k > \frac{n}{4}$. Using duality and essential self-adjointness of D_s, we see that $(1+D_s^2)^{-k} : L^1 \longrightarrow L^2$ is also a bounded map whose operator norm is bounded by a polynomial in s whenever $k > \frac{n}{4}$. Note that all of the statements above hold for the operator A as well as for D_s^2. Now, given a section $\omega \in \Gamma(M,E)^\rho$ supported on the complement of a 2ε-neighborhood of Crit(V) and $k > \frac{n}{4}$,

$$\begin{aligned}\|\phi(D_s)\,\omega\|_\infty &= \left\|\phi\left(\sqrt{A}\right)\omega\right\|_\infty \\ &\left\|(1+A)^{-k}(1+A)^k\,\phi\left(\sqrt{A}\right)(1+A)^k(1+A)^{-k}\,\omega\right\|_\infty \\ &\leq \left\|(1+A)^{-k}\right\|_{L^2\to L^\infty} c_k(s)\left\|(1+A)^{-k}\,\omega\right\|_2 \\ &\leq \left\|(1+A)^{-k}\right\|_{L^2\to L^\infty} c_k(s)\left\|(1+A)^{-k}\right\|_{L^1\to L^2}\|\omega\|_1 \\ &\leq p(s)\,c_k(s)\,\|\omega\|_1\end{aligned}$$

where $p(s)$ is a polynomial in s. Next, since ϕ is rapidly decreasing, $\phi(D_s)$ has a continuous kernel $k_s(x,y)$ (Lemma ???), and we have the inequality

$$\|k_s(x,\cdot)\|_\infty \leq \sup_{\int\|\omega\|=1,x\in M}\left\|\int_M k_s(x,y)\,\omega(y)\,\omega_{\text{vol}}(y)\right\| \leq p(s)\,c_k(s)$$

from the above. Thus, as $s \to \infty$, $k_s(x,y) \to 0$ uniformly for y in the complement of a 2ε -neighborhood of $\mathrm{Crit}(V)$ and $x \in M$. □

6. Local calculations for the equivariant index

6.1. **Index of the reduced transversal Dirac operator.** As in Section 2.3, suppose that the compact Lie group G acts on the closed, connected Riemannian manifold M of dimension n, where the dimensions of the orbits are not all the same. We assume that G acts by isometries. Let $\{V_1, ..., V_k\}$ be a set of vector fields on M induced from an orthonormal basis of $\mathfrak{g}$. Let E be a graded, self-adjoint $\mathbb{C}\mathrm{l}(TM)$ module over M, and let $c : TM \to \mathrm{Hom}(E,E)$ denote the Clifford multiplication. Let $D : \Gamma(M,E) \to \Gamma(M,E)$ be ordinary Dirac operator associated to this data. We consider the *reduced Dirac operator* $\widehat{D}$

$$\widehat{D} = D - \sum_{j=1}^{k} c(V_j)\nabla_{V_j}.$$

Next, let V be a fundamental vector field on M induced by an element $v \in \mathfrak{g}$. Let

$$\widehat{D}_s = \widehat{D} + isc(V).$$

Then

$$\widehat{D}_s^2 = \widehat{D}^2 + s\left(ic(V)\widehat{D} + i\widehat{D}c(V)\right) + s^2|V|^2.$$

Lemma 6.1. *With notation as above,*

$$\begin{aligned} \widehat{D}^2 &= \\ c(V)\widehat{D} + \widehat{D}c(V) &= \end{aligned}$$

Proof. Let $\{E_1, ..., E_n\}$ be a local framing of TM near a point $x \in M$.

$$\begin{aligned} \widehat{D}^2 &= \left(D - \sum_{j=1}^{k} c(V_j)\nabla_{V_j}\right)^2 \\ &= \left(\sum_{m=1}^{n} c(E_m)\nabla_{E_m} - \sum_{j=1}^{k} c(V_j)\nabla_{V_j}\right)^2 \\ &= D^2 - \sum_{m=1}^{n}\sum_{j=1}^{k} c(E_m)c(\nabla_{E_m}V_j)\nabla_{V_j} \\ &\quad - \sum_{m=1}^{n}\sum_{j=1}^{k} c(E_m)c(V_j)\nabla_{E_m}\nabla_{V_j} - \sum_{m=1}^{n}\sum_{j=1}^{k} c(V_j)c\left(\nabla_{V_j}E_m\right)\nabla_{E_m} \\ &\quad - \sum_{m=1}^{n}\sum_{j=1}^{k} c(V_j)c(E_m)\nabla_{V_j}\nabla_{E_m} + \sum_{j=1}^{k}\sum_{l=1}^{k} c(V_j)c\left(\nabla_{V_j}V_l\right)\nabla_{V_l} \\ &\quad + \sum_{j=1}^{k}\sum_{l=1}^{k} c(V_j)c(V_l)\nabla_{V_j}\nabla_{V_l}. \end{aligned}$$

Simplifying,

$$\begin{aligned}
\widehat{D}^2 &= D^2 - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(E_m\right) c\left(\nabla_{E_m} V_j\right) \nabla_{V_j} \\
&\quad - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(E_m\right) c\left(V_j\right) \nabla_{E_m}\nabla_{V_j} - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(V_j\right) c\left(\nabla_{V_j} E_m\right) \nabla_{E_m} \\
&\quad - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(V_j\right) c\left(E_m\right) \nabla_{E_m}\nabla_{V_j} - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(V_j\right) c\left(E_m\right) \nabla_{[E_m,V_j]} - \sum_{m=1}^{n}\sum_{j=1}^{k} c\left(V_j\right) c\left(E_m\right) R\left(E_m, V_j\right) \\
&\quad + \sum_{j=1}^{k}\sum_{l=1}^{k} c\left(V_j\right) c\left(\nabla_{V_j} V_l\right) \nabla_{V_l} \\
&\quad + \sum_{j=1}^{k}\sum_{l=1}^{k} c\left(V_j\right) c\left(V_l\right) \nabla_{V_j}\nabla_{V_l}.
\end{aligned}$$

Note that $\nabla_{V_j} = \mathcal{L}_{V_j} + A_j$, where $\mathcal{L}_{V_j}$ denotes the Lie derivative in direction V_j and A_j is an endomorphism of E. Also, L_{V_j} commutes with the Dirac operator D. We substitute

$$\begin{aligned}
\widehat{D}^2 &= \left(D - \sum_{j=1}^{k} c\left(V_j\right) \nabla_{V_j}\right)^2 \\
&= \left(\sum_{m=1}^{n} c\left(E_m\right) \nabla_{E_m} - \sum_{j=1}^{k} c\left(V_j\right) \mathcal{L}_{V_j} - \sum_{j=1}^{k} c\left(V_j\right) A_j\right)^2 \\
&= D^2 + \left(\sum_{j=1}^{k} c\left(V_j\right) \mathcal{L}_{V_j}\right)^2 + \left(\sum_{j=1}^{k} c\left(V_j\right) A_j\right)^2 \\
&\quad - \sum_{j=1}^{k} \left(c\left(V_j\right) D + D \circ c\left(V_j\right)\right) \mathcal{L}_{V_j} - \sum_{j=1}^{k} \left(D \circ c\left(V_j\right) A_j + c\left(V_j\right) A_j D\right) \\
&\quad + \sum_{j=1}^{k}\sum_{l=1}^{k} \left(c\left(V_j\right) c\left(\mathcal{L}_{V_j} V_l\right) A_l + c\left(V_j\right) c\left(V_l\right) \left(\mathcal{L}_{V_j} A_l\right)\right) \\
&\quad + \sum_{j=1}^{k}\sum_{l=1}^{k} \left(c\left(V_j\right) c\left(V_l\right) A_l + c\left(V_l\right) A_l c\left(V_j\right)\right) \mathcal{L}_{V_j}.
\end{aligned}$$

Thus

$$\begin{aligned}\widehat{D}^2 &= D^2 + \left(\sum_{j=1}^{k} c(V_j)\,\mathcal{L}_{V_j}\right)^2 + \left(\sum_{j=1}^{k} c(V_j)\,A_j\right)^2 \\ &\quad -\sum_{j=1}^{k}\left(c(V_j)\,D + D\circ c(V_j)\right)\mathcal{L}_{V_j} - \sum_{j=1}^{k}\left(D\circ c(V_j)\,A_j + c(V_j)\,A_j D\right) \\ &\quad +\sum_{j=1}^{k}\sum_{l=1}^{k}\left(c(V_j)\,c([V_j,V_l])\,A_l + c(V_j)\,c(V_l)\left(\mathcal{L}_{V_j}A_l\right)\right) \\ &\quad +\sum_{j=1}^{k}\sum_{l=1}^{k}\left(c(V_j)\,c(V_l)\,A_l + c(V_l)\,A_l c(V_j)\right)\mathcal{L}_{V_j}.\end{aligned}$$

From the Appendix, $c(V)\circ D + D\circ c(V) = -2\nabla_V - \operatorname{div}(V) + c(d(V^*))$, so that

$$\begin{aligned}c(V)\circ D + D\circ c(V) &= -2\nabla_V + c(d(V^*)) \\ &= -2\mathcal{L}_V - 2A_V + c(d(V^*))\end{aligned}$$

if V is a fundamental vector field, and if A_V is the endomorphism defined by

$$\nabla_V = \mathcal{L}_V + A_V.$$

Thus,

$$\begin{aligned}\widehat{D}^2 &= D^2 + \left(\sum_{j=1}^{k} c(V_j)\,\mathcal{L}_{V_j}\right)^2 + \left(\sum_{j=1}^{k} c(V_j)\,A_j\right)^2 \\ &\quad -\sum_{j=1}^{k}\left(-2\mathcal{L}_{V_j} - 2A_j + cd(V^*)\right)\mathcal{L}_{V_j} - \sum_{j=1}^{k}\left(D\circ c(V_j)\,A_j + c(V_j)\,A_j D\right) \\ &\quad +\sum_{j=1}^{k}\sum_{l=1}^{k}\left(c(V_j)\,c([V_j,V_l])\,A_l + c(V_j)\,c(V_l)\left(\mathcal{L}_{V_j}A_l\right)\right) \\ &\quad +\sum_{j=1}^{k}\sum_{l=1}^{k}\left(c(V_j)\,c(V_l)\,A_l + c(V_l)\,A_l c(V_j)\right)\mathcal{L}_{V_j}.\end{aligned}$$

□

7. Appendix

7.1. **Proof of lemma ??.**

Proof. We have

$$(D_s)^2 - D^2 = s\,(ic(V)\circ D + iD\circ c(V)) + s^2\,|V|^2\,.$$

Write $V = \sum V_j e_j$ in terms of a local orthonormal, isochronous frame $e_1, e_2, ...$ of the tangent bundle, corresponding to geodesic normal coordinate vector fields $e_j = \partial_j$ at the origin of the coordinate system. At the origin of the coordinate system, we have

$$\begin{aligned} ZD+DZ &= ic(V)\circ D + iD\circ c(V) \qquad (7.1)\\ &= i\sum_{j,k} V_j c(e_j)\,c(e_k)\,\nabla_k + i\sum_{j,k} c(e_k)\,\nabla_k \circ V_j c(e_j)\,, \text{ and since } \{e_j\} \text{ is isochronous}\\ &= i\sum_{j,k} V_j c(e_j)\,c(e_k)\,\nabla_k + i\sum_{j,k} V_j c(e_k)\,c(e_j)\,\nabla_k + i\sum_{j,k} c(e_k)\,c(e_j)\,\partial_k V_j\\ &= -2i\sum_{j} V_j\nabla_j - i\sum_j \partial_j V_j + i\sum_{j\neq k} c(e_k)\,c(e_j)\,\partial_k V_j, \text{ since } \{c(e_j),c(e_k)\}=\delta_{jk}\\ &= -2i\nabla_V - i\left(\operatorname{div}(V) - \sum_j V_j \sum_{k\neq j}\langle \nabla_k e_j, e_k\rangle\right) + i\sum_{j\neq k} c(e_k)\,c(e_j)\,\partial_k V_j\\ &= -2i\nabla_V - i\operatorname{div}(V) + i\sum_{j\neq k} c(e_k)\,c(e_j)\,\partial_k V_j \text{ since } \{e_j\} \text{ is isochronous}\\ &= -2i\nabla_V - i\operatorname{div}(V) + ic(d(V^*))\,, \end{aligned}$$

where by $c(d(V^*))$ we imply that we have used the inverse of the symbol map σ to convert the two-form $d(V^*)$ to a Clifford algebra element. For example, $\sigma(e_1e_2) = c(e_1)\,c(e_2)\,1 = (dx_1\wedge - dx_1\lrcorner)\,(dx_2\wedge -dx_2\lrcorner)\,1 = dx_1\wedge dx_2$, so we define $c(dx_1\wedge dx_2) = c(e_1e_2)$ at the origin. Now, since the last expression is coordinate-free, we conclude that

$$ZD+DZ = -2i\nabla_V - i\operatorname{div}(V) + ic(d(V^*))$$

at all points. ☐

7.2. **Proof of lemma ??.**

Proof. A Killing vector field X can be lifted to a vector field $\overline{X}$ on the frame bundle, so that $\overline{X}$ covers X and is invariant under the $SO(n)$ bundle. The vector field $\overline{X}$ lifts uniquely to a vector field $\widehat{X}$ on the principal spin bundle $\widetilde{F}$. Thus it acts on any bundle associated to $\widetilde{F}$, such as the spin bundle. Let X be an infinitesimal isometry. If g is the metric tensor, then $\mathcal{L}_X g = 0$. If Y, Z are any two tensor fields of the same type, then

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \mathcal{L}_X Y, Z\rangle + \langle Y, \mathcal{L}_X Z\rangle\\ &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\,. \end{aligned}$$

Thus $A_X = \mathcal{L}_X - \nabla_X$ is skew-symmetric and of degree zero, since $\langle A_X Y, Z\rangle = -\langle Y, A_X Z\rangle$. Hence its action on $\Gamma(TM)$ comes from the endomorphism (also called A_X) of TM. Choose a basis $\{e_i\}$ of T_xM, and we may identify A_X with the element $a_X \in \mathfrak{o}(n)$ by identifying T_xM with $\mathbb{R}^n$ using the basis . Under this identification antisymmetric matrix $a_X = \left((a_X)_{ij}\right)$corresponds to an endomorphism $A_X = \frac{1}{4}\sum (a_X)_{ij}\, e_i e_j$ (see Lemma 4.8 in [42] for calculations).

Let $\lambda : \mathrm{Spin}(n) \to SO(n)$ be the double cover, and let $d\lambda : \mathfrak{spin}(n) \to \mathfrak{o}(n)$ be the differential map on the Lie algebras. Observe that $\mathfrak{spin}(n) \cong \mathrm{Cl}_2(\mathbb{R}^n)$, and the Lie bracket induced on $\mathrm{Cl}_2(\mathbb{R}^n)$ is $[a,b] = ab - ba$ (using Clifford multiplication). For all $v \in \mathbb{R}^n$, $z \in \mathrm{Spin}(n)$,

$$d\lambda(z)(v) = zv - vz,$$

where z is thought of as an element of $\mathrm{Cl}_2(\mathbb{R}^n)$ and v is thought of as an element of $\mathrm{Cl}_1(\mathbb{R}^n)$. Hence

$$\begin{aligned} d\lambda\left(\frac{1}{4}\sum (a_X)_{ij}\, e_ie_j\right)(v) &= \left[\frac{1}{4}\sum (a_X)_{ij}\, e_ie_j, v\right] \\ &= a_X v, \end{aligned}$$

so that

$$A_X = \mathcal{L}_X - \nabla_X = \frac{1}{4}\sum (a_X)_{ij}\, e_ie_j.$$

Next, given a Killing field X and vector field Y,

$$\begin{aligned} A_XY &= \mathcal{L}_XY - \nabla_XY \\ &= [X,Y] - [X,Y] - \nabla_YX \\ &= -\nabla_YX. \end{aligned}$$

Thus, given any vector field Z,

$$\langle A_XY, Z\rangle = -\langle \nabla_YX, Z\rangle,$$

so $A_X = -(\nabla X)^{\#}$. This implies

$$(a_X)_{ij} = -\langle \nabla_{e_i}X, e_j\rangle.$$

Thus,

$$\begin{aligned} A_X &= \mathcal{L}_X - \nabla_X = -\frac{1}{4}\sum \langle \nabla_{e_i}X, e_j\rangle\, e_ie_j \\ &= -\frac{1}{4}\sum e_i\left(\langle \nabla_{e_i}X, e_j\rangle\, e_j\right) \\ &= -\frac{1}{4}\sum e_i\left(\nabla_{e_i}X\right) \\ &= -\frac{1}{4}\sum e_i\left(\partial_iX_j\right)e_j \text{ if } \{e_j\} \text{ is isochronous} \\ &= -\frac{1}{4}c\left(d\left(X^*\right)\right) \end{aligned}$$

We have therefore that

$$\mathcal{L}_X = \nabla_X - \frac{1}{4}c\left(d\left(X^*\right)\right) \tag{7.2}$$

if X is a Killing vector field. □

7.3. **Proof of lemma ??.**

Proof. First we will write down all the formulas necessary in our calculations. Let $\{e_i\}$be a synchronous framing at $p \in M$, then $[\nabla_V, c(e_i)] = 0$ at p for each i. In this framing we have the following well-known formulas:

$$[\mathcal{L}_V, \nabla_Y] = \nabla_{\mathcal{L}_VY} = \nabla_{[V,Y]}, \tag{7.3}$$

$$\mathcal{L}_Ve_i = \sum_j [V_je_j, e_i] = -\sum_j (\partial_iV_j)\, e_j \text{ at } p, \tag{7.4}$$

and

$$[V, e_i] = \nabla_Ve_i - \nabla_iV = -\nabla_iV. \tag{7.5}$$

In addition, we can combine the formula $D\circ c(V)+c(V)\circ D=-2iL_V+\frac{i}{2}c(d(V^*))$ (see (??)) with (??) to obtain

$$\mathcal{L}_V=\frac{1}{2}\nabla_V-\frac{1}{4}\left(D\circ c(V)+c(V)\circ D\right). \tag{7.6}$$

Then

$$\begin{aligned}
\mathcal{L}_V\circ D-D\circ\mathcal{L}_V &= \sum \mathcal{L}_V\circ\left(c(e_i)\nabla_i\right)-\left(c(e_i)\nabla_i\right)\circ\mathcal{L}_V \\
&= \sum\left[\mathcal{L}_V,c(e_i)\right]\circ\nabla_i+c(e_i)\nabla_{\mathcal{L}_Ve_i} \text{ by the formula (7.3)} \\
&= \sum_{i,j}\left[\left(\nabla_V-\frac{1}{4}c(e_j)c(\nabla_jV)\right),c(e_i)\right]\nabla_i+\sum_i c(e_i)\nabla_{\mathcal{L}_Ve_i} \\
&= \sum_{i,j,k}-\frac{1}{4}(\partial_jV_k)\left[c(e_j)c(e_k),c(e_i)\right]\nabla_i+\sum_i c(e_i)\nabla_{\mathcal{L}_Ve_i} \\
&= \sum_{i,j}\frac{1}{2}(\partial_jV_i)c(e_j)\nabla_i-\sum_{i,k}\frac{1}{2}(\partial_iV_k)c(e_k)\nabla_i+\sum_i c(e_i)\nabla_{\mathcal{L}_Ve_i} \\
&= \sum_{i\neq j}\frac{1}{2}(\partial_jV_i-\partial_iV_j)c(e_j)\nabla_i+\sum_i c(e_i)\nabla_{\mathcal{L}_Ve_i} \\
&= \sum_{i\neq j}\frac{1}{2}(\partial_jV_i-\partial_iV_j)c(e_j)\nabla_i-\sum_{i,j}(\partial_iV_j)c(e_i)\nabla_j \\
&= -\sum_{i\neq j}\frac{1}{2}(\partial_jV_i+\partial_iV_j)c(e_j)\nabla_i-\sum_i(\partial_iV_i)c(e_i)\nabla_i
\end{aligned}$$

By the definition of the Killing field $\mathcal{L}_V(g)=0$. Thus

$$\begin{aligned}
0 &= \mathcal{L}_V\left(g(e_i,e_j)\right) \\
&= \mathcal{L}_V(g)(e_i,e_j)+g\left(\mathcal{L}_V(e_i),e_j\right)+g\left(e_i,\mathcal{L}_V(e_j)\right) \\
&= g\left([V,e_i],e_j\right)+g\left(e_i,[V,e_j]\right) \\
&= -\partial_iV_j-\partial_jV_i.
\end{aligned}$$

Thus for a Killing field at the origin of synchronous frame $\partial_iV_j+\partial_jV_i=0$ and $\partial_iV_i=0$. Now we see that

$$\mathcal{L}_V\circ D-D\circ\mathcal{L}_V=-\sum_{i\neq j}\frac{1}{2}(\partial_jV_i+\partial_iV_j)c(e_j)\nabla_i-\sum_i(\partial_iV_i)c(e_i)\nabla_i=0.$$

Now let us check that $\mathcal{L}_V$ commutes with $c(V)$. In our computation we used the fact that Lie derivative with respect to a Killing field could be lifted to a Lie derivative on the spin bundle by $\mathcal{L}_V\circ c=c\circ\mathcal{L}_V$. The operator $\mathcal{L}_V$ commutes with H_s because it commutes with both D and $c(V)$. ∎

7.4. **Miscellaneous computations.** Next, for future reference, we compute the commutators $[D,\nabla_V]$ and $[c(V),\nabla_V]$. Using the same coordinate system as above,

$$\begin{aligned}
[D,\nabla_V] &= D\circ\nabla_V-\nabla_V\circ D\\
&= \sum_{j,k} c(e_k)\nabla_k\circ V_j\nabla_j - V_j\nabla_j\circ c(e_k)\nabla_k, \text{ so by isochronicity}\\
&= \sum_{j,k}(\partial_k V_j)\,c(e_k)\nabla_j + V_j c(e_k)\nabla_k\nabla_j - V_j c(e_k)\nabla_j\nabla_k\\
&= \sum_{j,k}(\partial_k V_j)\,c(e_k)\nabla_j + V_j c(e_k)(\nabla_k\nabla_j-\nabla_j\nabla_k)\\
&= \sum_{j,k}(\partial_k V_j)\,c(e_k)\nabla_j + c(V\lrcorner K),
\end{aligned}$$

where K is the curvature two-form. Also,

$$\begin{aligned}
[c(V),\nabla_V] &= c(V)\nabla_V-\nabla_V\circ c(V)\\
&= \sum_{j,k} V_k V_j c(e_k)\nabla_j - V_j\nabla_j\circ V_k c(e_k)\\
&= -\sum_{j,k} V_j(\partial_j V_k)\,c(e_k)
\end{aligned}$$

Also, we compute D^2: (same isochronous coordinate system)

$$\begin{aligned}
D^2 &= \sum_{j,k} c(e_k)\nabla_k\circ c(e_j)\nabla_j\\
&= \sum_{j,k} c(e_k)\,c(e_j)\nabla_k\nabla_j\\
&= -\sum_j \nabla_j^2+\sum_{k<j} c(e_k)\,c(e_j)(\nabla_k\nabla_j-\nabla_j\nabla_k)\\
&= -\sum_j \nabla_j^2+\sum_{k<j} c(e_k)\,c(e_j)\left(\nabla_k\nabla_j-\nabla_j\nabla_k-\nabla_{[e_k,e_j]}\right)\\
&= -\sum_j \nabla_j^2+\sum_{k<j} c(e_k)\,c(e_j)\,K(e_k,e_j)\\
&= -\sum_j \nabla_j^2+c(K),
\end{aligned}$$

where K is the curvature two-form.

Suppose that V is a Killing vector field. Then $\operatorname{div}(V)=0$, and ∇_V commutes with D and with $Z=ic(V)$. Thus, $D^2+s^2Z^2=D^2+s^2|V|^2$ and $ZD+DZ$ are simultaneously diagonalizable. So, given an eigenspace $E_n\subset\Gamma E$ of $ZD+DZ$, the restriction of D_s to E_n localizes at the singularities of the vector field V.

In general, we can do this if $ZD+DZ=A+B$, where A is first order, B is zeroth order, and $[A,D]=[A,B]=[A,Z]=0$. Thus, the equations read $[ZD+DZ-B,D]=$

$[ZD + DZ - B, B] = [ZD + DZ - B, Z] = 0$, or

$$\begin{aligned}
(ZD + DZ - B)\,D - D\,(ZD + DZ - B) &= ZD^2 - D^2Z - BD + DB = 0\\
(ZD + DZ - B)\,B - B\,(ZD + DZ - B) &= ZDB + DZB - BZD - BDZ = 0\\
(ZD + DZ - B)\,Z - Z\,(ZD + DZ - B) &= DZ^2 - Z^2D - BZ + ZB = 0.
\end{aligned}$$

References

[1] M. F. Atiyah, *Elliptic operators and compact groups*, Lecture Notes in Math. **401**, Springer-Verlag, Berlin, 1974.

[2] M. F. Atiyah, F. Hirzebruch, *Spin-manifolds and group actions.* 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 18–28 Springer, New York

[3] V. Belfi, E. Park, and K. Richardson, *A Hopf index theorem for foliations*, Differential Geom. Appl. **18** (2003), 319-341.

[4] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften **298**, Springer-Verlag, Berlin, 1992.

[5] N. Berline and M. Vergne, *The Chern character of a transversally elliptic symbol and the equivariant index*, Invent. Math. **124**(1996), no. 1-3, 11-49.

[6] N. Berline and M. Vergne, *L'indice équivariant des opérateurs transversalement elliptiques*, Invent. Math. **124** (1996), no. 1-3, 51-101.

[7] J.-M. Bismut, *The Witten complex and the degenerate Morse inequalities*, J. Differ. Geometry **23** (1986), 207-240.

[8] J.-M. Bismut and S. Goette, *Holomorphic equivariant analytic torsions*, Geom. Funct. Anal. **10** (2000), no. 6, 1289–1422.

[9] J.-M. Bismut and S. Goette, *Equivariant de Rham torsions*, preprint.

[10] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller. With an appendix by François Laudenbach*, Astérisque No. **205** (1992).

[11] J.-M. Bismut, W. Zhang, *Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle.* Geom. Funct. Anal. **4** (1994), no. 2, 136–212.

[12] R. Bott, *Morse theory indomitable*, Inst. Hautes Études Sci. Publ. Math. No. **68** (1988), 99–114.

[13] R. Bott, C. Taubes, *On the rigidity theorems of Witten*, J. Amer. Math. Soc. **2** (1989), no. 1, 137–186.

[14] R. Bott and L. Tu, *Differential forms in algebraic topology*, Springer-Verlag, New York, 1982.

[15] M. Braverman, *Holomorphic Morse inequalities and symplectic reduction*, Topology **38** (1999), no. 1, 71–78.

[16] M. Braverman, *Index theorem for equivariant Dirac operators on noncompact manifolds*, K-Theory **27** (2002), no. 1, 61–10.

[17] M. Braverman, *New proof of the Cobordism Invariance of the Index*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1095-1101.

[18] M. Braverman, Appendix J in V. Guillemin, V. Ginsburg, and Y. Karshon, *Moment maps, cobordisms, and hamiltonian group actions.* Mathematical surveys and monographs, **98**. American mathematical society, Providence, RI, 2002.

[19] M. Braverman, M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros*, Math. Proc. Cambridge Philos. Soc. **122** (1997), no. 2, 357–375.

[20] M. Braverman, M. Farber, *Equivariant Novikov inequalities* , K-Theory **12** (1997), no. 4, 293–318.

[21] D. Burghelea, *Lectures on Witten-Helffer-Sjöstrand theory* , Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997). Gen. Math. **5** (1997), 85–99..

[22] D. Burghelea and S. Haller, *On the topology and analysis of a closed one form. I (Novikov's theory revisited).* Essays on geometry and related topics, Vol. 1, 2, 133–175, Monogr. Enseign. Math., **38**, Enseignement Math., Geneva, 2001.

[23] P. R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations.* J. Functional Analysis **12** (1973), 401–414.

[24] M. Craioveanu, M. Puta, and T. Rassias, *Old and new aspects in spectral geometry. Mathematics and its Applications*, **534** . Kluwer Academic Publishers, Dordrecht, 2001.

[25] H. L. Cycon, R. G. Froese, W. Kirsh, and B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer-Verlag.
[26] J. Dieudonné, *Treatise on Analysis VIII*, Academic Press, Boston 1993
[27] A. El Soufi and X. P. Wang, *Some remarks on Witten's method. Poincaré-Hopf theorem and Atiyah-Bott formula*, Ann. Global Anal. Geom. **5** (1987), no. 3, 161–178.
[28] B. V. Fedosov, *Index theorems*. Partial differential equations, VIII, 155–251, Encyclopaedia Math. Sci., 65, Springer, Berlin, 1996
[29] H. Feng, E. Guo, *Novikov-type inequalities for vector fields with non-isolated zero points*, Pacific J. Math. **201** (2001), no. 1, 107–120.
[30] H. Feng, E. Guo, *A Hopf index theorem for a real vector bundle*, Chinese Ann. Math. Ser. B **23** (2002), no. 4, 507–518.
[31] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Mathematics, **1336**. Springer-Verlag, Berlin, 1988. vi+107 pp.
[32] B. Helffer and J. Sjöstrand, *Puits multiples en mecanique semi-classique, IV. Etude du complexe de Witten*, Commun. in Partial Differ. Equations **10**(3), 245-340 (1985).
[33] G. Henniart, *Les inégalités de Morse (d'après E. Witten)*, Seminar Bourbaki, Vol. 1983/84. Astérisque No. **121-122** (1985), 43–61.
[34] Z. Jianwei, *A note on the degenerate Morse inequalities*, Math. J. Okayama Univ. **42** (2000), 153–159.
[35] H. B. Lawson, Jr. and M-L. Michelsohn, *Spin Geometry* , Princeton Mathematical Series **38**, Princeton University Press, Princeton, 1989.
[36] O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985 (trans. by J. Lohwater).
[37] L. I. Nicolaescu, *Lectures on the geometry of manifolds*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[38] V. Nistor, *On the kernel of the equivariant Dirac operator*, Ann. Global Anal. Geom. **17** (1999), no. 6, 595–613.
[39] P.-E. Paradan, *Spin_c quantization and the K-multiplicities of the discrete series*, preprint, 2001, math.DG/0103222.
[40] I. Prokhorenkov, *Morse-Bott functions and the Witten Laplacian*, Commun in Analysis and Geometry, **7**(4), 841-918, 1999.
[41] I. Prokhorenkov, K. Richardson, Perturbations of Dirac operators, Preprint arXiv:math.DG/0307251 v2 Jul. 2003.
[42] J. Roe, *Elliptic operators, topology, and asymptotic methods*, Pitman Research Notes in Math. **179**, Longman Scientific and Technical, Harlow, 1988.
[43] M. A. Shubin, *Semiclassical asymptotics on covering manifolds and Morse inequalities*, Geom. Funct. Anal. **6** (1996), no. 2, 370–409.
[44] M. A. Shubin, *Novikov inequalities for vector fields* , The Gelfand Mathematical Seminars, 1993–1995, 243–274, Gelfand Math. Sem., Birkhäuser Boston, Boston, MA, 1996.
[45] M. A. Shubin, *Pseudodifferential operators and spectral theory.* Translated from the 1978 Russian original by Stig I. Andersson. Second edition. Springer-Verlag, Berlin, 2001.
[46] M. E. Taylor, *Partial Differential Equations I*, Springer-Verlag, New York, 1996.
[47] H.-K.Wai, *Witten-Helffer-Sjöstrand theory for S^1-equivariant cohomology*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2141–2182.
[48] E. Witten, *Supersymmetry and Morse Theory*, J. Differ. Geometry, **17**, 661-692 (1982).
[49] E. Witten, *Index of Dirac operators*, Quantum fields and strings: a course for mathematicians, Vol. 1, 475–511, Amer. Math. Soc., Providence, RI, 1999.
[50] E. Witten, *The index of the Dirac operator in loop space*. Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 161–181, Lecture Notes in Math., **1326**, Springer, Berlin, 1988
[51] S. Wu, *Equivariant holomorphic Morse inequalities. II. Torus and non-abelian group actions.* J. Differential Geom. **51** (1999), no. 3, 401–429.
[52] S. Wu, W. Zhang, *Equivariant holomorphic Morse inequalities. III. Non-isolated fixed points*, Geom. Funct. Anal. **8** (1998), no. 1, 149–178.

Department of Mathematics, Texas Christian University, Box 298900, Fort Worth, Texas 76129
E-mail address: i.prokhorenkov@tcu.edu, k.richardson@tcu.edu