SUSPENSION FOLIATIONS: INTERESTING EXAMPLES OF TOPOLOGY AND GEOMETRY

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Abstract. We give examples of foliations on suspensions and comment on their topological and geometric properties

1. Idea of foliation by suspension

Here is the simplest example of foliation by suspension. Let X be a manifold of dimension q, and let $f : X \to X$ be a bijection. Then we define the suspension $M = S^1 \times_f X$ as the quotient of $[0,1] \times X$ by the equivalence relation $(1, x) \sim (0, f(x))$.

$$M = S^1 \times_f X = [0,1] \times X \diagup \sim$$

Then automatically M carries two foliations: $\mathcal{F}_2$ consisting of sets of the form $F_{2,t} = \{(t,x)_\sim : x \in X\}$ and $\mathcal{F}_1$ consisting of sets of the form $F_{2,x_0} = \{(t,x) : t \in [0,1], x \in \mathcal{O}_{x_0}\}$, where the orbit $\mathcal{O}_{x_0}$ is defined as

$$\mathcal{O}_{x_0} = \left\{..., f^{-2}(x_0), f^{-1}(x_0), x_0, f(x_0), f^2(x_0), ...\right\},$$

where the exponent refers to the number of times the function f is composed with itself. Note that $\mathcal{O}_{x_0} = \mathcal{O}_{f(x_0)} = \mathcal{O}_{f^{-2}(x_0)}$, etc., so the same is true for F_{1,x_0}. Understanding the foliation $\mathcal{F}_1$ is equivalent to understanding the dynamics of the map f. If the manifold X is already foliated, you can use the construction to increase the codimension of the foliation, as long as f maps leaves to leaves.

The first set of examples concerns foliations of a map from the circle to itself.

Example A: Let $X = S^1$, let α be a fixed real number, and let $f : S^1 \to S^1$ be defined by $f(z) = e^{i\alpha} z$. The $S^1 \times_f S^1$ is topologically the 2-torus. It is a cylinder with the two ends identified with a twist. Note that if α is a rational multiple of 2π, then all of the leaves are closed. If α is irrational, then all of the leaves are dense. This is called a Kronecker foliation. Note that all leaves have no holonomy.

Example B: Let $X = S^1$, let $f : S^1 \to S^1$ be defined by $f(z) = \overline{z}$. The $S^1 \times_f S^1$ is topologically the Klein bottle. It is a cylinder with the two ends identified with a reflection. Observe that all leaves are closed — two of them have $\mathbb{Z}_2$ holonomy, and the others have trivial holonomy.

The next example is a codimension-2 foliation on a 3-manifold.

Example C: (This one is from [8] and [9].) Consider the one-dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere S^2. On S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1-z^2)}\cos\theta$, $y' = \sqrt{(1-z^2)}\sin\theta$, $z' = z$. Let α be an irrational multiple

Date: August, 2009.
1991 *Mathematics Subject Classification.* 53C12, 58G11, 58G18, 58G25.

of 2π, and let the three-manifold $M = [0,1] \times S^2 / \sim$, where $(1, z, \theta) \sim (0, z, \theta + \alpha)$. Here the function $f : S^2 \to S^2$ is

$$f(z, \theta) = (z, \theta + \alpha).$$

Endow M with the product metric on $T_{z,\theta,t}M \cong T_{z,\theta}S^2 \times T_t\mathbb{R}$. Let the foliation $\mathcal{F} = \mathcal{F}_1$ be defined by the immersed submanifolds $L_{z,\theta} = \cup_{n\in\mathbb{Z}} [0,1] \times \{z\} \times \{\theta + \alpha\}$ (not unique in θ). The leaf closures $\overline{L}_z$ for $|z| < 1$ are two-dimensional, and the closures corresponding to the poles ($z = \pm 1$) are one-dimensional. The basic functions are functions of z alone.

Example D: This foliation is a suspension of an irrational rotation of S^1 composed with an irrational rotation of S^2 on the manifold $S^1 \times S^2$. As in Example **??**, on S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1-z^2)}\cos\theta$, $y' = \sqrt{(1-z^2)}\sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let β be any irrational number. We consider the four-manifold $M = [0,1] \times S^2 \times [0,1] / \sim$, where $(0, z, \theta, t) \sim (1, z, \theta, t)$, $(1, z, \theta, s) \sim (0, z, \theta + \alpha, s + \beta \bmod 1)$. Let the foliation $\mathcal{F} = \mathcal{F}_1$ be defined by the immersed submanifolds $L_{z,\theta,s} = \cup_{n\in\mathbb{Z}} [0,1] \times \{z\} \times \{\theta + \alpha\} \times \{s + \beta\}$ (not unique in θ or s). The leaf closures $\overline{L}_z$ for $|z| < 1$ are three-dimensional, and the closures corresponding to the poles ($z = \pm 1$) are two-dimensional.

The following two examples are related to an example in [2]. The first is a codimension two foliation that does not admit a Riemannian foliation structure, and the second is a codimension two Riemannian foliation that is not taut.

Example E: Consider the flat torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Consider the map $F : T^2 \to T^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod 1$$

Let $M = [0,1] \times T^2 / \sim$, where $(1, a) \sim (0, F(a))$. Let $\mathcal{F}_1$ be the foliation whose leaves are of the form $L_a = \{(t,p)_\sim \in M : t \in [0,1], p \in \mathcal{O}_a\}$. This is an example of an Anosov foliation.

Example F: Consider the flat torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Consider the map $F : T^2 \to T^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod 1$$

Let $M = [0,1] \times T^2 / \sim$, where $(1, a) \sim (0, F(a))$. Let v, v' be orthonormal eigenvectors of the matrix above, corresponding to the eigenvalues $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$, respectively. Let the linear foliation $\mathcal{F}$ be defined by the vector v' on each copy of T^2.

2. More general suspensions

The most general type of suspension is as follows. Let Y be a manifold with fundamental group $\pi_1(Y)$ and universal cover $\widetilde{Y}$, let X be another manifold, and let $\phi : \pi_1(Y) \to \mathrm{Maps}(X)$, where by Maps we mean some group of bijective maps from X to itself, such as continous maps, smooth maps, analytic maps, isometries, etc. Then we define the suspension $M = Y \times_\phi X$ by

$$M = Y \times_\phi X = \widetilde{Y} \times X / \pi_1(Y),$$

where $g \in \pi_1(Y)$ acts on $\widetilde{Y} \times X$ by $g(\widetilde{y}, x) = (\widetilde{y} \cdot g^{-1}, \phi(g)(x))$, where $\widetilde{y}$ is mapped to $\widetilde{y} \cdot g^{-1}$ by the deck transformation corresponding to $g^{-1} \in \pi_1(Y)$. Note that this quotient is the same as the quotient by the equivalence relation $(\widetilde{y} \cdot g, x) \sim (\widetilde{y}, \phi(g)(x))$. The foliations of the suspension M come from choosing the immersed Y-parameter submanifolds or the immersed X-parameter submanifolds, or other submanifolds of these submanifolds. The standard foliation to choose is the foliation $\mathcal{F}_1$ of Y-parameter submanifolds, that is sets of the form $L_x = \left\{ [(\widetilde{y}, x')] : \widetilde{y} \in \widetilde{Y}, x' \in \mathcal{O}_x \right\}$, where $\mathcal{O}_x = \{\phi(g)(x) : g \in \pi_1(Y)\} \subset X$. Note that this generalizes the previous section, where in that section $Y = S^1 = [0,1] \diagup (0 = 1)$, $\pi_1(Y) = \mathbb{Z}$, and the homomorphism is $\phi(n)(x) = f^n(x)$.

Remark 2.1. *If you have any discrete, finitely-generated group G of bijective maps on X, there always exists a closed manifold Y and a homomorphism $\phi : \pi_1(Y) \to G$. This follows from the fact that every such group can be realized as the fundamental group of a closed 4-manifold, where ϕ can be then taken to be the identity. In general one may usually take Y to be simpler. For example if $\{g_1, g_2, g_3\}$ generates G, then one could take Y to be the connected sum of three copies of $S^2 \times S^1$, which has fundamental group the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, and the homomorphism could be generated by $\phi(n_j) = g_j^{n_j}$ in the group, for $j = 1, 2, 3$.*

Remark 2.2. *The choice of group of maps determines the transverse type of foliation $\mathcal{F}_1$. If the homomorphism ϕ maps to isometries of X, then $\mathcal{F}_1$ is a Riemannian foliation. If ϕ maps to morphisms of a Kähler manifold X, then $\mathcal{F}_1$ is a transversely Kähler foliation.*

We now give two examples of these more general suspensions.

The following example is a codimension two transversally oriented Riemannian foliation in which all the leaf closures have codimension one, and the leaf closure foliation is not transversally orientable. There are two leaf closures with $\mathbb{Z}_2$ holonomy.

Example G: This foliation is the suspension of an irrational rotation of the flat torus and a $\mathbb{Z}_2$-action. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ — the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = \widetilde{X} \times S^1 \times S^1 \diagup \pi_1(X)$, where $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by $\alpha(\theta, \phi) = (2\pi - \theta, 2\pi - \phi)$ and $\beta(\theta, \phi) = \left(\theta, \phi + \sqrt{2}\pi\right)$ on $S^1 \times S^1$. We use the standard product-type metric. The leaves of $\mathcal{F}$ are defined to be sets of the form $\left\{ (x, \theta, \phi)_\sim \,|\, x \in \widetilde{X} \right\}$. Note that the foliation is transversally oriented. The leaf closures are sets of the form

$$\overline{L}_\theta = \left\{ (x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi] \right\} \bigcup \left\{ (x, 2\pi - \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi] \right\}.$$

The next example is a codimension two Riemannian foliation with dense leaves, such that some leaves have holonomy but most do not.

Example H: This Riemannian foliation is a suspension of a pair of rotations of the sphere S^2. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ — the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = \widetilde{X} \times S^2 \diagup \pi_1(X)$. The group $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by rotations on S^2 in the following ways. Thinking of S^2 as imbedded in $\mathbb{R}^3$, let α act by an irrational rotation around the z-axis, and let β act by an irrational rotation around the x-axis. We use the standard product-type metric. As usual, the leaves of $\mathcal{F}$ are defined to be sets of the form

$\left\{(x,v)_\sim \,|\, x\in \widetilde{X}\right\}$. Note that the foliation is transversally oriented, and a generic leaf is simply connected and thus has trivial holonomy. Also, the every leaf is dense. The leaves $\{(x,(1,0,0))_\sim\}$ and $\{(x,(0,0,1))_\sim\}$ have nontrivial holonomy; the closures of their infinitessimal holonomy groups are copies of $SO(2)$.

3. Cohomology

3.1. **Basic Cohomology.** Note that the basic forms are the smooth forms on the whole manifold M that only depend locally on the transverse coordinates (if one chooses a foliation chart with coordinates adapted to the foliation. In other words, a form $\beta\in\Omega\left(M\right)$ is **basic** for the foliation $\mathcal{F}$ if $i\left(X\right)\beta=0$ and $i\left(X\right)d\beta=0$ for all vectors X tangent to the leaves, i.e. $X\in T\mathcal{F}$. Here, $i\left(X\right)$ means **interior product** with the vector X, a pointwise operator that is linear and that depends linearly on X. If α is a k-form, then $i\left(X\right)\alpha$ is the $(k-1)$-form defined by $i\left(X\right)\alpha\left(v_1,...,v_{k-1}\right)=\alpha\left(X,v_1,...,v_{k-1}\right)$. In local coordinates, if for instance $\alpha=\alpha\left(y\right)dy_1\wedge...\wedge dy_k$, then $i\left(\frac{\partial}{\partial y_j}\right)\alpha\left(y\right)dy_1\wedge...\wedge dy_k=\left(-1\right)^{j-1}\alpha\left(y\right)dy_1\wedge...\wedge\widehat{dy_j}\wedge...\wedge dy_k$, where the $\widehat{\cdot}$ means that term is omitted. Let $\Omega\left(M,\mathcal{F}\right)$ denote the space of basic forms, and let $\Omega^k\left(M,\mathcal{F}\right)$ denote the space of basic k-forms.

From the definition, we see that since $d^2=0$, if β is basic, then also $d\beta$ is basic. (Proof: if β is basic and $\alpha=d\beta$, then for all $X\in T\mathcal{F}$ we have $i\left(X\right)\alpha=i\left(X\right)d\beta=0$, and $i\left(X\right)d\alpha=i\left(X\right)d^2\beta=i\left(X\right)0=0$.) Hence

$$d^k:=d:\Omega^k\left(M,\mathcal{F}\right)\to\Omega^{k+1}\left(M,\mathcal{F}\right)$$

with $d^2=0$. We may take real or complex-valued functions. We define the **basic cohomology groups** to be the quotient groups (or quotient vector spaces, since each $\Omega^k\left(M,\mathcal{F}\right)$ is a vector space (infinite-dimensional) and d is a linear transformation) defined by

$$H_b^k\left(M,\mathcal{F}\right)=\frac{\ker d^k}{\operatorname{Im} d^{k-1}}.$$

One may also think of these as topological vector spaces with the quotient topology, giving first $\Omega^k\left(M,\mathcal{F}\right)$ the smooth topology (i.e. as a Fréchet space - Ok don't go there).

3.2. **Leafwise cohomology.** Consider the bigrading on the set of all forms as follows. Given any Riemannian metric on M, let $T\mathcal{F},N\mathcal{F}\subset\mathcal{TM}$ denote the tangent and normal bundles of the foliation, and let $T^*\mathcal{F},N^*\mathcal{F}\subset T^*M$ denote the cotangent and conormal bundles of the foliation. Observe that only $T\mathcal{F}$ and $N^*\mathcal{F}$ may be defined independent of the metric; the other two bundles mentioned depend on the choice of metric. Let $\wedge^{i,j}T^*M=\wedge^iN^*\mathcal{F}\otimes\wedge^jT^*\mathcal{F}\subset\wedge^{i+j}T^*M$ be a bigrading of forms at a point, so that

$$\begin{aligned}\wedge^kT^*M &= \bigoplus_{i+j=k}\wedge^{i,j}T^*M\\ \Omega^k\left(M\right) &= \Gamma\left(\wedge^kT^*M\right)=\bigoplus_{i+j=k}\Omega^{i,j}\left(M,\mathcal{F}\right).\end{aligned}$$

If we choose a local orthonormal frame $\left(e_1,...,e_q,e_{q+1},...,e_{p+q}\right)$ of TM such that $N\mathcal{F}=\operatorname{span}\left\{e_1,...,e_q\right\}$, $T\mathcal{F}=\operatorname{span}\left\{e_{q+1},...,e_{p+q}\right\}$, we see that

$$\wedge^{i,j}T^*M=\operatorname{span}\left\{e_{k_1}^*\wedge...\wedge e_{k_i}^*\wedge e_{l_1}^*\wedge...\wedge e_{l_q}^*:1\leq k_1<...<k_i\leq q\text{ and }q+1\leq l_1<...<l_j\leq p+q\right\}.$$

Note that the differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has the property that for a general orthonormal frame (e_j),

$$de_p^* = -\sum \Gamma_{ij}^{\,p} e_i^* \wedge e_j^*,$$

where $\Gamma_{ij}^{\,p}$ are the Christoffel symbols defined by $\nabla_{e_i} e_j = \sum \Gamma_{ij}^{\,p} e_p$. Thus we expect the differential to split into four possible parts:

$$\begin{aligned} d &: \Omega^{i,j}(M,\mathcal{F}) \to \Omega^{i+2,j-1}(M,\mathcal{F}) \oplus \Omega^{i+1,j}(M,\mathcal{F}) \oplus \Omega^{i,j+1}(M,\mathcal{F}) \oplus \Omega^{i-1,j+2}(M,\mathcal{F}) \\ d &= d_{2,-1} + d_{1,0} + d_{0,1} + d_{-1,2}. \end{aligned}$$

To keep track of indices, let Roman indices (i, j, k, etc) refer to the leafwise vectors, and let the Greek indices (α, β, γ, etc) refer to the the normal vectors. Due to the integrability condition $[e_i, e_j] \subset T\mathcal{F}$, we have

$$de_\alpha^*(e_i, e_j) = e_i(e_\alpha^*(e_j)) - e_j(e_\alpha^*(e_i)) - e_\alpha^*([e_i, e_j]) = 0 - 0 - 0,$$

so that $d_{-1,2} = 0$ always. Thus,

$$d = d_{2,-1} + d_{1,0} + d_{0,1},$$

$\Omega^{i+2,j-1}\oplus$and where $d_{2,-1}$ is also zero if and only if the normal bundle is integrable. Since $d^2 = 0$, we see that since d^2 maps $\Omega^{i,j}$ to $\Omega^{i+4,j-2} \oplus \Omega^{i+3,j-1} \oplus \Omega^{i+2,j} \oplus \Omega^{i+1,j+1} \oplus \Omega^{i,j+2}$, each piece must be zero, so that

$$\begin{aligned} d_{2,-1}^2 &= 0 \\ d_{2,-1}d_{1,0} + d_{1,0}d_{2,-1} &= 0 \\ d_{1,0}^2 + d_{2,-1}d_{0,1} + d_{0,1}d_{2,-1} &= 0 \\ d_{1,0}d_{0,1} + d_{0,1}d_{1,0} &= 0 \\ d_{0,1}^2 &= 0 \end{aligned}$$

The last differential is called the leafwise derivative: we let $d_{\mathcal{F}} = d_{0,1}$, and we usually restrict this to leafwise differential forms, that is elements of $\Omega^{0,*}(M,\mathcal{F})$. The resulting **leafwise cohomology groups** (or topological vector spaces) are

$$H_{\mathcal{F}}^k(M) = \frac{\ker d_{\mathcal{F}}^k}{\operatorname{Im} d_{\mathcal{F}}^{k-1}},$$

where

$$d_{\mathcal{F}}^k = d_{0,1} : \Omega^{0,k}(M,\mathcal{F}) \to \Omega^{0,k+1}(M,\mathcal{F}).$$

One can use the bigrading and such differentials to produce a spectral sequence for the foliation.

There are many other types of cohomology groups associated to foliations.

3.3. **Remarks about these cohomology groups.** Many of the facts about standard de Rham cohomology do not hold for these more general kinds of cohomology theories. For example, the dimensions of the cohomology spaces can be infinite, and the topologies on these spaces do not have to be Hausdorff. Further, one does not usually have a foliation version of Poincare duality.

4. Molino theory

Let M be an n-dimensional, closed, connected, oriented Riemannian manifold without boundary, and let $\mathcal{F}$ be a transversally–oriented, codimension q foliation on M for which the metric is bundle–like. Let $\widehat{M}$ be the oriented transverse orthonormal frame bundle of $(M,\mathcal{F})$, and let p be the natural projection $p:\widehat{M}\longrightarrow M$. The manifold $\widehat{M}$ is a principal $SO(q)$-bundle over M. Given $\hat{x}\in\widehat{M}$, let $\hat{x}g$ denote the well-defined right action of $g\in SO(q)$ applied to $\hat{x}$. Associated to $\mathcal{F}$ is the lifted foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$. The lifted foliation is transversally parallelizable, and the closures of the leaves are fibers of a fiber bundle $\widehat{\pi}:\widehat{M}\longrightarrow\widehat{W}$. The manifold $\widehat{W}$ is smooth and is called the basic manifold (see [6, pp. 105-108, p. 147ff]). Let $\overline{\widehat{\mathcal{F}}}$ denote the foliation of $\widehat{M}$ by leaf closures of $\widehat{\mathcal{F}}$.

$$\begin{array}{ccccccc} \widetilde{E} & & & & p^*E & & \\ & \searrow & & & \downarrow & & \\ & & \widehat{W} & \overset{\widehat{\pi}}{\longleftarrow} & \left(\widehat{M},\widehat{\mathcal{F}}\right) & \hookleftarrow & SO\left(q\right) \\ & & \downarrow & \circlearrowleft & \downarrow^{p} & & \\ & & W & \longleftarrow & (M,\mathcal{F}) & & \end{array}$$

Endow $\widehat{M}$ with the Sasakian metric $g^M+g^{SO(q)}$, where g^M is the pullback of the metric on M, and $g^{SO(q)}$ is the standard, normalized, biinvariant metric on the fibers. By this, we mean that we use the transverse Levi–Civita connection (see [6, p. 80ff]) to do the following. We calculate the inner product of two horizontal vectors in $T_{\hat{x}}\widehat{M}$ by using g^M, and we calculate the inner product of two vertical vectors using $g^{SO(q)}$. We require that vertical vectors are orthogonal to horizontal vectors. This metric is bundle–like for both $(\widehat{M},\widehat{\mathcal{F}})$ and $(\widehat{M},\overline{\widehat{\mathcal{F}}})$. The transverse metric on $(\widehat{M},\overline{\widehat{\mathcal{F}}})$ induces a well–defined Riemannian metric on $\widehat{W}$. The group $G=SO(q)$ acts by isometries on $\widehat{W}$ according to $\widehat{\pi}(\hat{x})g:=\widehat{\pi}(\hat{x}g)$ for $g\in SO(q)$.

For each leaf closure $\overline{\widehat{L}}\in\overline{\widehat{\mathcal{F}}}$ and $\widehat{x}\in\overline{\widehat{L}}$, the restricted map $p:\overline{\widehat{L}}\to\overline{L}$ is a principle bundle with fiber isomorphic to a subgroup $H_{\widehat{x}}<SO(q)$, which is the isotropy subgroup at the point $\widehat{\pi}(\hat{x})\in\widehat{W}$. The conjugacy class of this group is an invariant of the leaf closure $\overline{L}$, and the number of different dimensions of these groups is the number of different dimensions of leaf closures of $(M,\mathcal{F})$.

5. The mean curvature form and basic Laplacian

We assume $(M,\mathcal{F},g_M)$ is a Riemannian foliation with bundle-like metric compatible with the Riemannian structure $(M,\mathcal{F},g_Q)$. For later use, we define the mean curvature one-form κ and discuss the operator $\kappa_b\lrcorner$. Let

$$H=\sum_{i=1}^{p}\pi\left(\nabla^M_{f_i}f_i\right),$$

where $\pi:TM\to N\mathcal{F}$ is the bundle projection and $(f_i)_{1\leq i\leq p}$ is a local orthonormal frame of $T\mathcal{F}$. This is the mean curvature vector field, and its dual one-form is $\kappa=H^\flat$. Let $P:L^2\left(\Omega\left(M\right)\right)\to L^2\left(\Omega_b\left(M,\mathcal{F}\right)\right)$ be the L^2-orthogonal projection of all forms onto basic forms. Let $\kappa_b=P\kappa$ be the basic projection of this mean curvature one-form. In the case of a bundle-like metric, this form is smooth and calculated from κ by averaging over the leaf closures (see [7]). It turns out that κ_b is a closed form whose cohomology class in $H_b^1(M,\mathcal{F})$ is independent of the choice of bundle-like metric (see [1]).

The easiest way to calculate κ is to use **Rummler's formula**. If $\chi_{\mathcal{F}}$ is the leafwise volume form $\chi_{\mathcal{F}} = e_{q+1}^* \wedge ... \wedge e_{p+q}^* \in \Omega^{0,p}$ or **characteristic form** of the foliation, then we have the formula

$$d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0,$$

where $\varphi_0 \in \Omega^{2,p-1}$ measures the lack of integrability of the normal bundle. We then see that

$$\kappa = (-1)^{p+1} \chi_{\mathcal{F}} \lrcorner d\chi_{\mathcal{F}},$$

where $\chi_{\mathcal{F}}\lrcorner$ means the (pointwise) adjoint of the wedge product operator $\chi_{\mathcal{F}}\wedge$.

We have the following expression for δ_b, the L^2-adjoint of d restricted to the space of basic forms of a particular degree (see [10], [7]):

$$\begin{aligned}\delta_b &= P\delta \\ &= \pm \bar{*} d \bar{*} + \kappa_b \lrcorner \\ &= \delta_T + \kappa_b \lrcorner,\end{aligned}$$

where

- δ_T is the formal adjoint (with respect to g_Q) of the exterior derivative on the transverse local quotients.
- the pointwise transversal Hodge star operator $\bar{*}$ is defined on all k-forms γ by

  $$\bar{*}\gamma = (-1)^{p(q-k)} * (\gamma \wedge \chi_{\mathcal{F}}),$$

  with $\chi_{\mathcal{F}}$ being the leafwise volume form, the characteristic form of the foliation and $*$ being the ordinary Hodge star operator. Note that $\bar{*}^2 = (-1)^{k(q-k)}$ on k-forms.
- The sign $\pm$ above only depends on dimensions and the degree of the basic form.

The **basic Laplacian** Δ_b corresponding to a bundle-like metric is defined to be

$$\Delta_b = d\delta_b + \delta_b d : \Omega_b^*(M,\mathcal{F}) \to \Omega_b^*(M,\mathcal{F}).$$

This operator and its spectrum depend on the choice of bundle-like metric. The kernel of the basic Laplacian consists of basic-harmonic forms, and these forms generate the basic cohomology (see [5], [7]).

The trace of the basic heat kernel on k-forms is

$$K_B(t) = \sum_{m\geq 0} e^{-\lambda_m t},$$

where $0 \leq \lambda_0 \leq \lambda_1 \leq ...$ are the eigenvalues of Δ_b restricted to $\Omega_b^k(M,\mathcal{F})$.

Note that in a recent paper [3], it is mentioned that the twisted differentials

$$\widetilde{d} = d - \frac{1}{2}\kappa_b\wedge, \widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b\lrcorner.$$

give a new basic Laplacian

$$\widetilde{\Delta_b} = \widetilde{d}\widetilde{\delta} + \widetilde{\delta}\widetilde{d}$$

whose spectrum does not depend on the choice of bundle-like metric (as long as the transverse metric is fixed). Also, the basic cohomology groups $\widetilde{H_b^k}(M,\mathcal{F})$ corresponding to the differential $\widetilde{d}$ satisfy Poincaré duality on transversally oriented Riemannian foliations, and the dimensions of these cohomology groups are again independent of the choice of bundle-like metric.

6. Topological and geometric properties of examples

In this section I have repeated the definitions so the reader need not look back. A lot of these examples are in my papers [8], [9].

Example A: Let $X = S^1$, let α be a fixed real number, and let $f : S^1 \to S^1$ be defined by $f(z) = e^{i\alpha} z$. The $S^1 \times_f S^1$ is topologically the 2-torus. It is a cylinder with the two ends identified with a twist. Note that if α is a rational multiple of 2π, then all of the leaves are closed. If α is irrational, then all of the leaves of the horizontal foliation are dense. This is called a Kronecker foliation. Note that all leaves have no holonomy.

In the case where $\alpha = \frac{p}{q}(2\pi)$ with $\frac{p}{q}$ in lowest terms, let the torus be considered as $[0, 2\pi] \times [0, 2\pi]$ with the sides identified accordingly, each leaf consists of q horizontal lines, and the leaf space can be identified as the torus $\left[0, \frac{2\pi}{q}\right] \times [0, 2\pi]$. If α is an irrational multiple of 2π, then every leaf is dense. The flat metric is bundle-like for this foliation. Basic forms are $\{f(y) + g(y)\, dy\}$, where $(x, y) \in S^1 \times S^1$ are the coordinates of the foliation. If α is an irrational multiple of 2π, then f and g must be constant. If $\alpha = \frac{p}{q}(2\pi)$ as above, then we only must have that f and g are periodic with period $\frac{2\pi}{q}$. The basic cohomology group dimensions in both cases are $h_b^0 = 1 = h_b^1$. The leafwise cohomology groups are very interesting for this example. If α is an irrational multiple of 2π, then the leafwise cohomology groups can be infinite dimensional and actually can be nonHausdorff, depending on the type of irrational number $\frac{\alpha}{2\pi}$ is (whether it is *Liouville* or not). See [4].

Example B: Let $X = S^1$, let $f : S^1 \to S^1$ be defined by $f(z) = \overline{z}$. The $S^1 \times_f S^1$ is topologically the Klein bottle. It is a cylinder with the two ends identified with a reflection. Observe that all leaves of the horizontal foliation are closed — two of them have $\mathbb{Z}_2$ holonomy, and the others have trivial holonomy. Again the basic forms must be of the form $\{f(y) + g(y)\, dy : y \in S^1\}$, but note that $f(\overline{y}) = f(y),\ g(\overline{y}) = -g(y)$ are required. Every basic one-form is exact (because it integrates to zero), and so the basic cohomology betti numbers are $h_b^0 = 1,\ h_b^1 = 0$.

Example C: One-dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere S^2. On S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1 - z^2)} \cos\theta$, $y' = \sqrt{(1 - z^2)} \sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let the three-manifold $M = S^2 \times [0, 1] / \sim$, where $(z, \theta, 0) \sim (z, \theta + \alpha, 1)$. Endow M with the product metric on $T_{z,\theta,t} M \cong T_{z,\theta} S^2 \times T_t \mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta} = \cup_{n \in \mathbb{Z}} \{z\} \times \{\theta + \alpha\} \times [0, 1]$ (not unique in θ). The leaf closures $\overline{L}_z$ for $|z| < 1$ are two-dimensional, and the closures corresponding to the poles ($z = \pm 1$) are one-dimensional. This is a codimension-2 foliation on a 3-manifold. Here, $SO(2)$ acts on the basic manifold, which is homeomorphic to a sphere. In this case, the principal orbits have isotropy type $(\{e\})$, and the two fixed points obviously have isotropy type $(SO(2))$. In this example, the isotropy types correspond precisely to the infinitessimal holonomy groups.

The basic functions are functions of z alone, and the basic Laplacian on functions is $\Delta_B = -(1 - z^2)\, \partial_z^2 + 2z\, \partial_z$. The volume form on M is $dz\, d\theta\, dt$, and the volume of the leaf closure at z is $\frac{1}{2\pi\sqrt{1-z^2}}$ for $|z| < 1$. The eigenfunctions are the Legendre polynomials $P_n(z)$ corresponding to eigenvalues $m(m+1)$ for $m \geq 0$. From this information alone, one may calculate that the trace $K_B(t)$ of the basic heat operator is

$$K_B(t) = \sum_{m\geq 0} e^{-m(m+1)t} = \frac{1}{\sqrt{4\pi t}}\left(\pi + \frac{\pi}{4}t + O\left(t^2\right)\right).$$

The basic manifold $\widehat{W}$ corresponding to this foliation is a sphere with points described by orthogonal coordinates $(z, \varphi) \in [-1, 1] \times (-\pi, \pi]$. As shown in [8], the metric on $\widehat{W}$ is given by $\langle \partial_z, \partial_z \rangle = \frac{1}{1-z^2}$, $\langle \partial_\varphi, \partial_\varphi \rangle = \frac{4\pi^2\left(1-z^2\right)}{4\pi^2(1-z^2)+z^2}$.
Let's now calculate the Euler characteristic of this foliation. Since the foliation is taut, the standard Poincare-type duality works, and $H^0_B(M) \cong H^2_B(M) \cong \mathbb{R}$. It suffices to check the dimension h^1 of the cohomology group $H^1_B(M)$. Then the basic Euler characteristic is $\chi(M, \mathcal{F}) = 1 - h^1 + 1 = 2 - h^1$. Smooth basic functions are of the form $f(z)$, where $f(z)$ is smooth in z for $-1 < z < 1$ and is of the form $f(z) = f_1(1 - z^2)$ near $z = 1$ for a smooth function f_1 and is of the form $f(z) = f_2(1 - z^2)$ near $z = -1$ for a smooth function f_2. Smooth basic one-forms are of the form $\alpha = g(z)\,dz + k(z)\,d\theta$, where $g(z)$ and $k(z)$ are smooth functions for $-1 < z < 1$ and satisfy

$$\begin{aligned} g(z) &= g_1\left(1 - z^2\right) \text{ and} \\ k(z) &= \left(1 - z^2\right) k_1\left(1 - z^2\right) \end{aligned} \tag{6.1}$$

near $z = 1$ and

$$\begin{aligned} g(z) &= g_2\left(1 - z^2\right) \text{ and} \\ k(z) &= \left(1 - z^2\right) k_2\left(1 - z^2\right) \end{aligned}$$

near $z = -1$ for smooth functions g_1, g_2, k_1, k_2. A simple calculation shows that $\ker d^1 = \operatorname{im} d^0$, so that $h^1 = 0$. Thus, $\chi(M, \mathcal{F}) = 2$.
It is instructive to see how the trace of the heat kernel fits into this example. The basic Hodge star $\overline{*}$ (see either [5] or [7]) can be computed as follows:

$$\begin{aligned} \overline{*}dz &= \left(1 - z^2\right) d\theta \\ \overline{*}d\theta &= -\frac{1}{1 - z^2} dz \\ \overline{*}(dz \wedge d\theta) &= 1. \end{aligned}$$

We have already computed the asymptotics of the trace of the basic heat kernel on functions (and thus on two forms as well, since $\overline{*}$ commutes with the basic Laplacian in the taut case). We now compute the asymptotics of the trace of the basic heat operator on one-forms. The basic adjoint of d is $\delta_B = -\overline{*}d\overline{*}$ on both one-forms and two-forms, and we compute that

$$\begin{aligned} \delta_B(g(z)\,dz + k(z)\,d\theta) &= -\overline{*}d\overline{*}(g(z)\,dz + k(z)\,d\theta) \\ &= -\partial_z\left(\left(1 - z^2\right) g(z)\right) \\ \delta_B(h(z)\,dz \wedge d\theta) &= -\overline{*}d\overline{*}(h(z)\,dz \wedge d\theta) \\ &= -\left(1 - z^2\right) h'(z)\,d\theta. \end{aligned}$$

We then compute the basic Laplacian on one-forms:

$$\begin{aligned}\Delta_B\left(g\left(z\right)dz+k\left(z\right)d\theta\right) &= \left(\delta_B d+d\delta_B\right)\left(g\left(z\right)dz+k\left(z\right)d\theta\right)\\ &= -\left(\partial_z\right)^2\left(\left(1-z^2\right)g\left(z\right)\right)dz+\left(1-z^2\right)k''\left(z\right)d\theta.\end{aligned}$$

The resulting eigenvalue problem separates into two eigenvalue problems for $g(z)$ and $k(z)$. These are both special cases of the Jacobi differential equation; the eigenvalues for $g(z)$ are $(n+2)(n+1)$ for $n\geq 0$, and the eigenvalues for $k(z)$ are $(n-1)n$ for $n\geq 2$. In the latter case the two zero eigenvalues ($n=0,1$) had to be thrown out because the resulting eigenfunctions are not of the correct form (6.1). Thus, we have that

$$\operatorname{tr}\left(e^{-t\Delta_B^1}\right)=2\sum_{n\geq 1}e^{-n(n+1)t},$$

which by Equation **??** is

$$\begin{aligned}\operatorname{tr}\left(e^{-t\Delta_B^1}\right) &= 2\operatorname{tr}\left(e^{-t\Delta_B^0}\right)-2\\ &= \frac{1}{\sqrt{4\pi t}}\left(2\pi+\frac{\pi}{2}t+O\left(t^2\right)\right)-2.\end{aligned}$$

Thus, as expected, the supertrace of the basic Laplacian on forms is

$$\operatorname{tr}\left(e^{-t\Delta_B^0}\right)-\operatorname{tr}\left(e^{-t\Delta_B^1}\right)+\operatorname{tr}\left(e^{-t\Delta_B^2}\right)=2=\chi\left(M,\mathcal{F}\right).$$

Observe that in this case, the form of the asymptotic expansion for one forms is slightly different than that for functions. In particular, this example shows that the orbit space can be dimension 1 (odd) and yet have nontrivial index.

Example D: This foliation is a suspension of an irrational rotation of S^1 composed with an irrational rotation of S^2 on the manifold $S^1\times S^2$. As in Example **??**, on S^2 we use the cylindrical coordinates (z,θ), related to the standard rectangular coordinates by $x'=\sqrt{(1-z^2)}\cos\theta$, $y'=\sqrt{(1-z^2)}\sin\theta$, $z'=z$. Let α be an irrational multiple of 2π, and let β be any irrational number. We consider the four-manifold $M=S^2\times[0,1]\times[0,1]/\sim$, where $(z,\theta,0,t)\sim(z,\theta,1,t)$, $(z,\theta,s,0)\sim(z,\theta+\alpha,s+\beta \bmod 1,1)$. Endow M with the product metric on $T_{z,\theta,s,t}M\cong T_{z,\theta}S^2\times T_s\mathbb{R}\times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta,s}=\cup_{n\in\mathbb{Z}}\{z\}\times\{\theta+\alpha\}\times\{s+\beta\}\times[0,1]$ (not unique in θ or s). The leaf closures $\overline{L}_z$ for $|z|<1$ are three-dimensional, and the closures corresponding to the poles ($z=\pm 1$) are two-dimensional. This is a codimension-3 Riemannian foliation for which all of the infinitessimal holonomy groups are trivial; moreover, the leaves are all simply connected. There are leaf closures of codimension 2 and codimension 1. The codimension 2 leaf closures correspond to isotropy type (e) on the basic manifold, and the codimension 1 leaf closures correspond to an isotropy type $(SO(2))$ on the basic manifold. In some sense, the isotropy type measures the holonomy of the leaf closure in this case.

The basic forms in the various dimensions are:

$$\begin{aligned}\Omega_B^0 &= \left\{f\left(z\right)\right\}\\ \Omega_B^1 &= \left\{g_1\left(z\right)dz+\left(1-z^2\right)g_2(z)d\theta+g_3\left(z\right)ds\right\}\\ \Omega_B^2 &= \left\{h_1\left(z\right)dz\wedge d\theta+\left(1-z^2\right)h_2(z)d\theta\wedge ds+h_3\left(z\right)dz\wedge ds\right\}\\ \Omega_B^3 &= \left\{k\left(z\right)dz\wedge d\theta\wedge ds\right\},\end{aligned}$$

where all of the functions above are smooth in a neighborhood of $[0,1]$. An elementary calculation shows that $h^0 = h^1 = h^2 = h^3 = 1$, so that $\chi(M,\mathcal{F}) = 0$. It is pretty easy to generalize the calculations from the last example to get that the supertrace of the basic heat operator on forms is:

$$\begin{aligned}\chi(M,\mathcal{F}) &= \operatorname{tr}\left(e^{-t\Delta_B^0}\right) - \operatorname{tr}\left(e^{-t\Delta_B^1}\right) + \operatorname{tr}\left(e^{-t\Delta_B^2}\right) - \operatorname{tr}\left(e^{-t\Delta_B^3}\right) \\ &= \sum_{n\geq 0} e^{-n(n+1)t} + 3\sum_{n\geq 0} e^{-n(n+1)t} - 2 - \left(3\sum_{n\geq 0} e^{-n(n+1)t} - 2\right) + \sum_{n\geq 0} e^{-n(n+1)t} \\ &= 0,\end{aligned}$$

as expected. Note that taut foliations of odd codimension will always have a zero Euler characteristic, by Poincare duality. Open Question: will these foliations always have a zero basic index?

Example E: Consider the flat torus $T^2 = \mathbb{R}^2 \diagup \mathbb{Z}^2$. Consider the map $F : T^2 \to T^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod 1$$

Let $M = [0,1] \times T^2 \diagup \sim$, where $(1,a) \sim (0,F(a))$. Let $\mathcal{F}_1$ be the foliation whose leaves are of the form $L_a = \{(t,p)_\sim \in M : t \in [0,1], p \in \mathcal{O}_a\}$.

It can be shown that the basic cohomology H_b^1 is infinite-dimensional, because there is an infinite-dimensional space of closed basic one-forms, and the only basic functions are constants. The transversal volume form is exact, so $h_b^2 = 0$, $h_b^0 = 1$, $h_b^1 = \infty$. Therefore, there is no Riemannian foliation structure on this foliation, because if there were h_b^1 would have to be finite.

Example F: In ([2, p. 80ff]). Consider the flat torus $T^2 = \mathbb{R}^2 \diagup \mathbb{Z}^2$. Consider the map $F : T^2 \to T^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \bmod 1$$

Let $M = [0,1] \times T^2 \diagup \sim$, where $(0,a) \sim (1,F(a))$. Let v, v' be orthonormal eigenvectors of the matrix above, corresponding to the eigenvalues $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$, respectively. Let the linear foliation $\mathcal{F}$ be defined by the vector v' on each copy of T^2. Notice that every leaf is simply connected and that the leaf closures are of the form $\{t\} \times T^2$, and this foliation is Riemannian if we choose a suitable metric. For example, we choose the metric along $[0,1]$ to be standard and require each torus to be orthogonal to this direction. Then we define the vectors v and v' to be orthogonal in this metric and let the lengths of v and v' vary smoothly over $[0,1]$ so that $\|v\|(0) = \frac{3+\sqrt{5}}{2}\|v\|(1)$ and $\|v'\|(0) = \frac{3-\sqrt{5}}{2}\|v'\|(1)$. Let $\overline{v} = a(t)\,v$, $\overline{v'} = b(t)\,v'$ be the resulting renormalized vector fields. This foliation is a codimension two Riemannian foliation that is not taut.

The basic manifold is a torus, and the isotropy groups are all trivial. We use coordinates $(t,x,y) \in [0,1] \times T^2$ to describe points of M. The basic forms are:

$$\begin{aligned}\Omega_B^0 &= \{f(t)\} \\ \Omega_B^1 &= \{g_1(t)\,dt + g_2(t)\overline{v}^*\} \\ \Omega_B^2 &= \{h(t)dt \wedge \overline{v}^*\},\end{aligned}$$

where all the functions are smooth. Note that $d\overline{v}^* = -\frac{a'(t)}{a(t)}\, dt \wedge \overline{v}^*$ By computing the cohomology groups, we get $h^0 = h^1 = 1$, $h^2 = 0$. Thus, the basic Euler characteristic is zero. The calculation using the heat kernel is also interesting. The differentials and codifferentials are as follows:

$$\begin{aligned} d\left(f\left(t\right)\right) &= f'\left(t\right)dt \\ d\left(g_1\left(t\right)dt + g_2(t)\overline{v}^*\right) &= a\left(\frac{g_2}{a}\right)'\left(t\right)dt \wedge \overline{v}^* \\ \delta_B\left(g_1\left(t\right)dt + g_2(t)\overline{v}^*\right) &= -g_1'\left(t\right) \\ \delta_B\left(h\left(t\right)dt \wedge \overline{v}^*\right) &= -a\left(\frac{h}{a}\right)'\overline{v}^*. \end{aligned}$$

From this we obtain:

$$\begin{aligned} \Delta_B\left(f\left(t\right)\right) &= -f''\left(t\right) \\ \Delta_B\left(g_1\left(t\right)dt + g_2(t)\overline{v}^*\right) &= -g_1''\left(t\right)\ dt - a\left(\frac{g_2}{a}\right)''\left(t\right)\overline{v}^* \\ \Delta_B\left(h\left(t\right)\ dt \wedge \overline{v}^*\right) &= -a\left(\frac{h}{a}\right)''\left(t\right)\ dt \wedge \overline{v}^*. \end{aligned}$$

All the functions above are functions on $[0,1]$ with periodic boundary conditions. We get the following expansions for the trace of the basic heat operator on forms:

$$\begin{aligned} \operatorname{tr}\left(e^{-t\Delta_B^0}\right) &= 1 + 2\sum_{n\geq 1} e^{-4\pi^2 n^2 t} \sim \frac{1}{\sqrt{4\pi t}} \\ \operatorname{tr}\left(e^{-t\Delta_B^1}\right) &= \operatorname{tr}_{S^1}\left(e^{-tL}\right) + 1 + 2\sum_{n\geq 1} e^{-4\pi^2 n^2 t} \sim \frac{1}{\sqrt{4\pi t}}\left(1 + A_1 t + A_2 t^2 + \ldots\right) + \frac{1}{\sqrt{4\pi t}} \\ \operatorname{tr}\left(e^{-t\Delta_B^2}\right) &= \operatorname{tr}_{S^1}\left(e^{-tL}\right) \sim \frac{1}{\sqrt{4\pi t}}\left(1 + A_1 t + A_2 t^2 + \ldots\right), \end{aligned}$$

where L is the elliptic operator on functions on the circle of length one defined by $Lh = -a\left(\frac{h}{a}\right)''$. Clearly, the supertrace is identically zero, as predicted. Note that in this case the asymptotics of the basic heat operators have no t^0 terms.

Example G: This foliation is the suspension of an irrational rotation of the flat torus and a $\mathbb{Z}_2$-action. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$ — the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = \widetilde{X} \times S^1 \times S^1 \diagup \pi_1(X)$, where $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by $\alpha(\theta, \phi) = (2\pi - \theta, 2\pi - \phi)$ and $\beta(\theta, \phi) = \left(\theta, \phi + \sqrt{2}\pi\right)$ on $S^1 \times S^1$. We use the standard product-type metric. The leaves of $\mathcal{F}$ are defined to be sets of the form $\left\{(x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}\right\}$. Note that the foliation is transversally oriented. The leaf closures are sets of the form

$$\overline{L}_\theta = \left\{(x, \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi]\right\} \bigcup \left\{(x, 2\pi - \theta, \phi)_\sim \,|\, x \in \widetilde{X}, \phi \in [0, 2\pi]\right\}$$

This example is a codimension two transversally oriented Riemannian foliation in which all the leaf closures have codimension one. The leaf closure foliation is not transversally orientable, and the basic manifold is a flat Klein bottle with an $SO(2)$-action. The two leaf closures with $\mathbb{Z}_2$ holonomy correspond to the two orbits of type

$(\mathbb{Z}_2)$, and the other orbits have trivial isotropy.
The basic forms are:

$$\begin{aligned}\Omega_B^0 &= \{f(\theta)\} \\ \Omega_B^1 &= \{g_1(\theta)\, d\theta + g_2(\theta) d\phi\} \\ \Omega_B^2 &= \{h(\theta) d\theta \wedge d\phi\},\end{aligned}$$

where the functions are smooth and satisfy

$$\begin{aligned}f(2\pi - \theta) &= f(\theta) \\ g_i(2\pi - \theta) &= -g_i(\theta) \\ h(2\pi - \theta) &= h(\theta).\end{aligned}$$

A simple argument shows that $h^0 = h^2 = 1$ and $h^1 = 0$. Thus, $\chi(M, \mathcal{F}) = 2$. The basic manifold $\widehat{W}$ is an $SO(2)$-manifold, defined by $\widehat{W} = [0, \pi] \times S^1 \big/ \sim$, where the circle has length 1 and $(\theta = 0 \text{ or } \pi, \gamma) \sim (\theta = 0 \text{ or } \pi, -\gamma)$. This is a Klein bottle, since it is the connected sum of two projective planes. $SO(2)$ acts on $\widehat{W}$ via the usual action on S^1. It is a simple exercise to calculate the trace of the basic heat operators:

$$\begin{aligned}\operatorname{tr}\left(e^{-t\Delta_B^0}\right) &= \sum_{n \geq 0} e^{-n^2 t} \sim \frac{\sqrt{\pi}}{2} t^{-1/2} + \frac{1}{2} \\ \operatorname{tr}\left(e^{-t\Delta_B^1}\right) &= 2\sum_{n \geq 1} e^{-n^2 t} \sim \sqrt{\pi} t^{-1/2} - 1 \\ \operatorname{tr}\left(e^{-t\Delta_B^2}\right) &= \sum_{n \geq 0} e^{-n^2 t} \sim \frac{\sqrt{\pi}}{2} t^{-1/2} + \frac{1}{2}.\end{aligned}$$

The basic Euler class is again the supertrace.
Another interesting feature of this example is the following. One may calculate the heat kernel explicitly, and its asymptotics have some interesting features. Let's restrict to the case of functions for simplicity. The normalized eigenfunctions are

$$\left\{\frac{1}{2\pi}\right\} \cup \left\{\frac{1}{\sqrt{2}\pi} \cos n\theta\right\}_{n>0}$$

corresponding to the eigenvalues $\{n^2\}_{n\geq 0}$. Thus

$$\begin{aligned}
K_B(t,\theta,\theta) &= \frac{1}{4\pi^2}+\frac{1}{2\pi^2}\sum_{n>0}e^{-n^2t}\cos^2 n\theta \\
&= \frac{1}{4\pi^2}+\frac{1}{4\pi^2}\sum_{n>0}e^{-n^2t}(1+\cos 2n\theta) \\
&= \frac{1}{4\pi^2}+\frac{1}{4\pi^2}\sum_{n>0}e^{-n^2t}+\frac{1}{8\pi^2}\left(1+2\sum_{n>0}e^{-n^2t}\cos 2n\theta\right)-\frac{1}{8\pi^2} \\
&= \frac{1}{8\pi^2}+\frac{1}{4\pi^2}\sum_{n>0}e^{-n^2t}+\frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}}\left(\frac{\sqrt{\pi}}{\sqrt{t}}e^{-(n\pi+\theta)^2/t}\right) \\
&\sim \frac{1}{8\pi^2}+\frac{1}{4\pi^2}\left(\frac{\sqrt{\pi}}{2\sqrt{t}}-\frac{1}{2}\right)+\frac{1}{8\pi^2}\begin{cases}\frac{\sqrt{\pi}}{\sqrt{t}} & \text{if } \theta=\pi \text{ or } 0\\ 0 & \text{otherwise}\end{cases} \\
&\sim \begin{cases}\frac{1}{4\pi^{3/2}\sqrt{t}} & \text{if } \theta=\pi \text{ or } 0\\ \frac{1}{8\pi^{3/2}\sqrt{t}} & \text{otherwise}\end{cases}.
\end{aligned}$$

On the other hand, the trace of the heat kernel on functions is

$$\begin{aligned}
K_B(t) &= \int_M K_B(t,\theta,\theta) \\
&= \sum_{n\geq 0}e^{-n^2t} \\
&\sim \frac{\sqrt{\pi}}{2\sqrt{t}}+\frac{1}{2},
\end{aligned}$$

as we noted before. Note that the asymptotics of $K_B(t,\theta,\theta)$ are integrable but do not integrate to the asymptotics of $K_B(t)$, because the $\frac{1}{2}$ would be missing. (!!!)

Example H: This Riemannian foliation is a suspension of a pair of rotations of the sphere S^2. Let X be any closed Riemannian manifold such that $\pi_1(X)=\mathbb{Z}*\mathbb{Z}$ — the free group on two generators $\{\alpha,\beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M=\widetilde{X}\times S^2\diagup\pi_1(X)$. The group $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by rotations on S^2 in the following ways. Thinking of S^2 as imbedded in $\mathbb{R}^3$, let α act by an irrational rotation around the z-axis, and let β act by an irrational rotation around the x-axis. We use the standard product-type metric. As usual, the leaves of $\mathcal{F}$ are defined to be sets of the form $\left\{(x,v)_\sim \,|\, x\in\widetilde{X}\right\}$. Note that the foliation is transversally oriented, and a generic leaf is simply connected and thus has trivial holonomy. Also, the every leaf is dense. The leaves $\{(x,(1,0,0))_\sim\}$ and $\{(x,(0,0,1))_\sim\}$ have nontrivial holonomy; the closures of their infinitessimal holonomy groups are copies of $SO(2)$. Thus, a leaf closure in $\widehat{M}$ covering the leaf closure M has structure group $SO(2)$ and is thus all of $\widehat{M}$, so that $\widehat{W}$ is a point. This example is a codimension two Riemannian foliation with dense leaves, such that some leaves have holonomy but most do not. The basic manifold is a point, the fixed point set of the $SO(2)$ action. The isotropy group $SO(2)$ measures the holonomy of some of the leaves contained in the leaf closure.

The only basic forms are constants and 2-forms of the form CdV, where C is a constant and dV is the volume form on S^2. Thus $h^0 = h^2 = 1$ and $h^1 = 0$, so that $\chi(M, \mathcal{F}) = 2$. The heat kernel approach is pretty silly, since the only eigenvalue is zero:

$$\begin{aligned} \operatorname{tr}\left(e^{-t\Delta_B^0}\right) &= 1 \\ \operatorname{tr}\left(e^{-t\Delta_B^1}\right) &= 0 \\ \operatorname{tr}\left(e^{-t\Delta_B^2}\right) &= 1. \end{aligned}$$

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