# The eta invariant and the equivariant index theorem
joint work with J. Brüning, F. Kamber

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# Plan of the talk:

1. Vielen Dank to the organizers
2. Extremely brief history of equivariant index theory
3. Statement of the Equivariant Index Theorem
4. Explanation and examples of theorem ingredients
5. Sketch of main ideas in proof
6. Applications:
   - equivariant Euler characteristic
   - foliation index theorem

# Abridged History of Equivariant Index Theory

- Atiyah-Singer I, 1968: Require the use of equivariant index for K-theory proof of the Atiyah-Singer index theorem, expressing the index of an elliptic operator in terms of topological invariants.
- Atiyah-Bott, 1966; Atiyah-Segal II, 1968: Lefschetz formula for character of equivariant index of elliptic operator in terms of fixed point set of a particular element of the group.
- Atiyah, 1974 book: Properties of equivariant index for transversally elliptic operators. Formula for invariant index of transversally elliptic operators, if all isotropy groups have the same dimension.
- Atiyah-Patodi-Singer II, 1975: Circle action with isolated fixed points on a 4-manifold: formula for invariant index of signature operator in terms of isolated fixed points, involving eta invariant.

# Abridged History, continued

- Kawasaki, 1981: Orbifold index theorem, index given in terms of invariants at different strata of orbifold.
- Brüning-Heintze, 1978 and 1984: properties of equivariant, transversally elliptic operators and heat kernels.
- Berline-Vergne, 1996: Generalized Lefschetz formula for virtual character of equivariant index of transversally elliptic operator in terms of fixed point sets.
- Connes-Moscovici, 1998: Transverse Index Theorem in Noncommutative Geometry - different kind of equivariant index.

# Main theorem and corollaries

## Theorem

*Equivariant Index Theorem, BKR 2007*

$$\mathrm{ind}^{\rho}\left(D\right) = \int_{\widetilde{M_0}\diagup G} A_0^{\rho}\left(x\right)\ \widetilde{|dx|}$$
$$+ \frac{1}{2}\sum_{i,j,a,b} \frac{n_j^{\rho}\gamma_{ab}^{j}}{n_b^{\rho}} \left(-\eta\left(D_i^{S+,\sigma_a}\right) + h\left(D_i^{S+,\sigma_a}\right)\right) \int_{\widetilde{\Sigma_i}\diagup G} A_{i,\sigma_b^*}^{\rho 0}\left(x\right)\ \widetilde{|dx|}$$

## Corollary

*Invariant Index Theorem, BKR 2007*

$$\mathrm{ind}^{\rho_0}\left(D\right) = \int_{\widetilde{M_0}\diagup G} A_0^{\rho_0}\left(x\right)\ \widetilde{|dx|}$$
$$+ \frac{1}{2}\sum_{i,\sigma} \left(-\eta\left(D_i^{S+,\sigma}\right) + h\left(D_i^{S+,\sigma}\right)\right) \int_{\widetilde{\Sigma_i}\diagup G} A_{i,\sigma^*}^{\rho_0}\left(x\right)\ \widetilde{|dx|}$$

# Applications

## Corollary

*Equivariant Euler characteristic*

$$\chi^{\rho}(M) = \chi^{\rho}(G \diagup H_{\mathrm{pr}})\, \chi(G \diagdown M_0, G \diagdown \textit{lower strata}) + \frac{1}{2} \sum_{i,\ b} \left( n_b^{\rho} + n_{\bar{b}}^{\rho} \right) \chi(G \diagup H_i)^{\sigma_b} \chi(G \diagdown \Sigma_i, G \diagdown \textit{lower strata})$$

## Theorem

*Basic Index Theorem, BKR 2007*

$$\mathrm{ind}_b(D_b) = \int_{\widetilde{M_0} \diagup \overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \frac{1}{2} \sum_{i,\rho} \left( -\eta\left(D^{S+,\rho}\right) + h\left(D^{S+,\rho}\right) \right) \int_{\widetilde{M_i} \diagup \overline{\mathcal{F}}} A_{i,b}^{\rho^*}(x)\ \widetilde{|dx|},$$