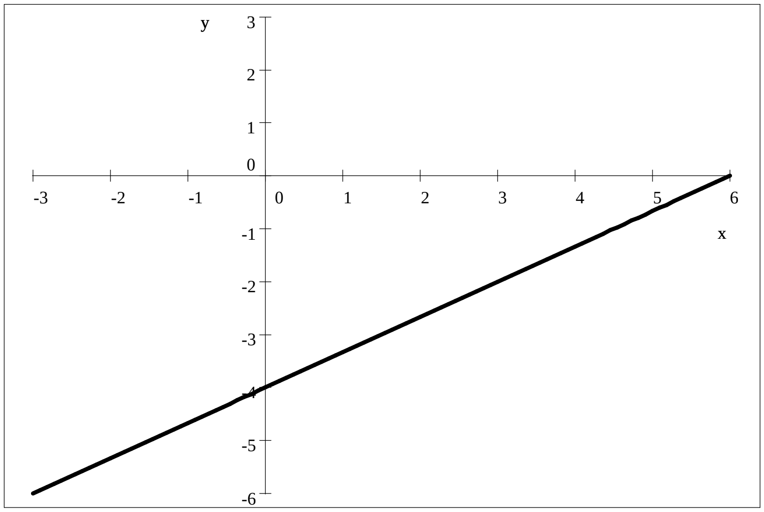
Curves

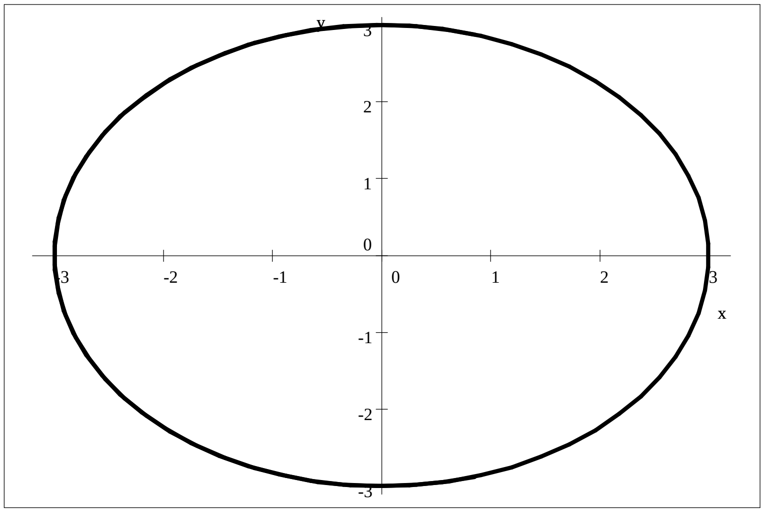
Let's look at some examples of plane curves $(x(t), y(t))$ in $\mathbb{R}^2$, the Euclidean plane, or space curves $(x(t), y(t), z(t))$ in $\mathbb{R}^3$, 3-dimensional space. We will always assume the functions $x(t)$ and $y(t)$ are infinitely differentiable.

See
http://www-groups.dcs.st-and.ac.uk/~history/Curves/Curves.html
for many more famous examples of plane curves.

Example *A straight line:* $\alpha(t) = (3t, 2t - 4)$. *That is,* $x(t) = 3t$, $y(t) = 2t - 4$.



Example *The circle of radius* 3*:* $\beta(t) = (3\cos(t), 3\sin(t))$.



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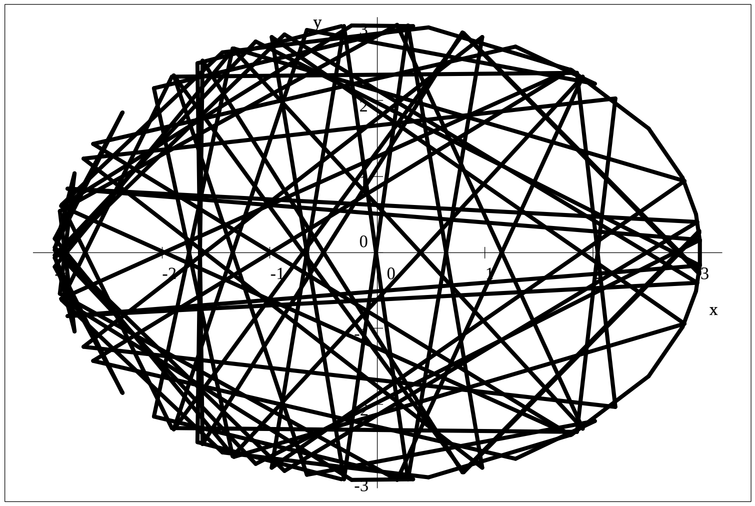
Example *Also the circle:* $c(t) = (3\cos(5t^3 + t), 3\sin(5t^3 + t))$.



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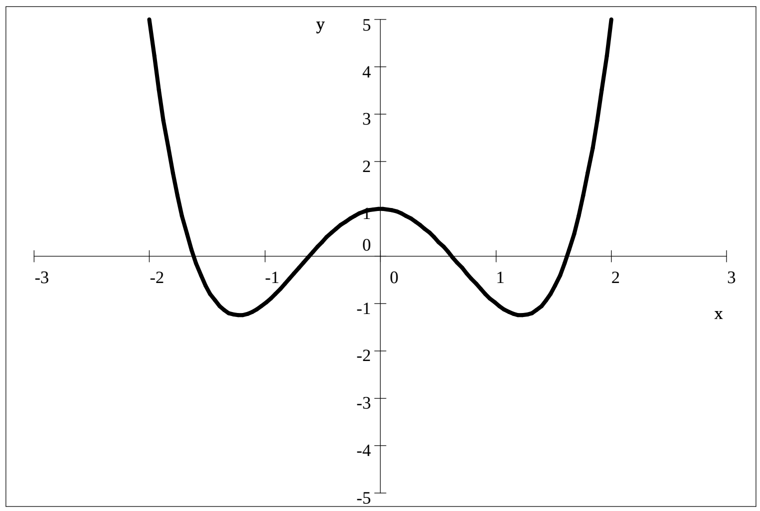
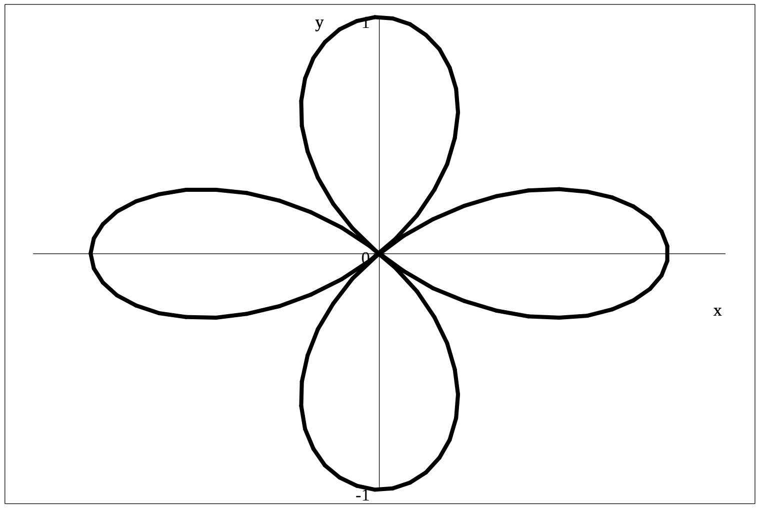
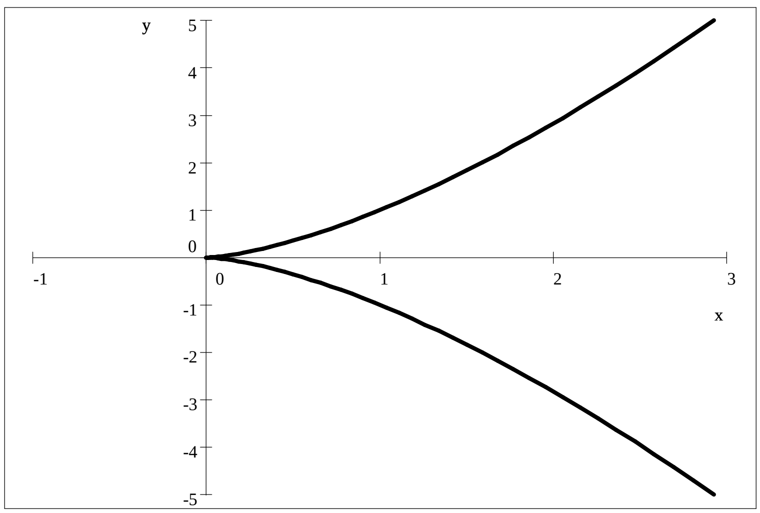
Example *The curve that is the graph of* $y = x^4 - 3x^2 + 1$: $\phi(t) = (t, t^4 - 3t^2 + 1)$.



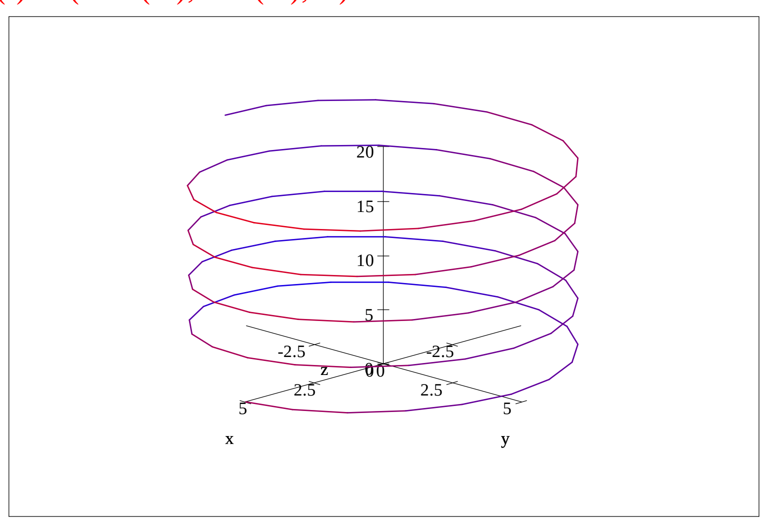
Example *(Double) Lemniscate of Bernoulli* $r^2 = |\cos(2\theta)|$:
$L(t) = (\cos(2t)\cos(t), \cos(2t)\sin(t))$



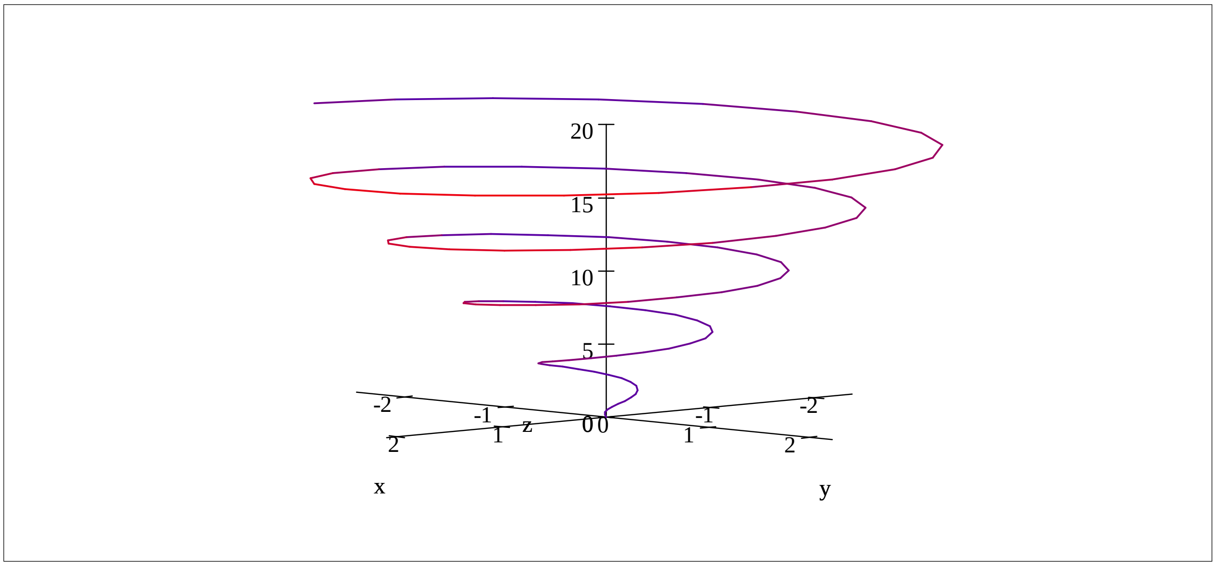
Example *Another plane curve* $K(t) = (t^2, t^3)$



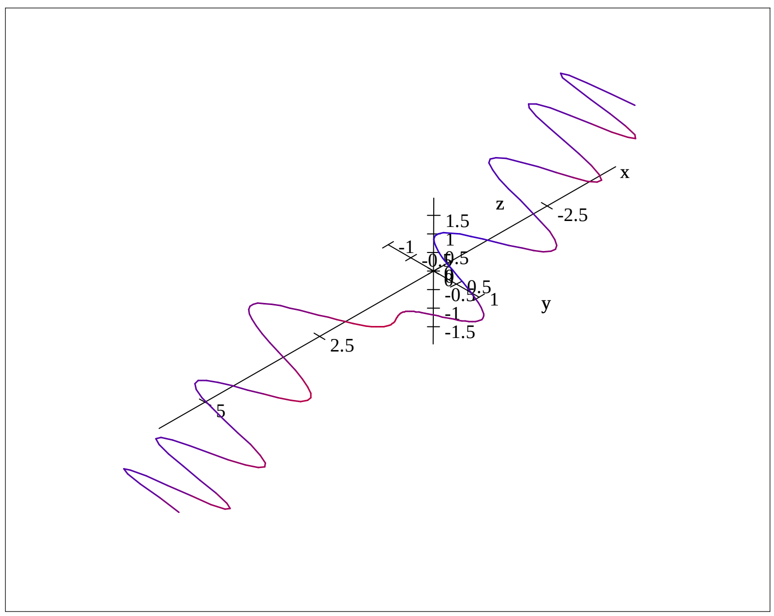
Example *A helix* $\eta(t) = (5\cos(3t), 5\sin(3t), 2t)$



Example *Slight modification of last example:* $\tilde{\eta}(t) = (5\frac{t}{20}\cos(3t), 5\frac{t}{20}\sin(3t), 2t)$.
Inside Hallesan curve:



Example *Something else* $\varphi(t) = (t+1, \cos(t^2-1), \sin(t) - \frac{t}{5})$



Velocity, Speed, Arclength, Tangent Vector Field

The **velocity** of the directed curve α is $\dot{\alpha} = \alpha'(t) = (x'(t), y'(t), z'(t))$. This is a vector that starts at $\alpha(t)$ and is tangent to α. The **speed** of α is the function $|\alpha'(t)|$, the magnitude (length) of the velocity vector. That is,

$$\boxed{\textbf{speed} = |\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.}$$

Using the speed we can calculate the **arclength** $s(t)$ between $\alpha(0)$ and $\alpha(t)$ along the curve α.

Since distance = (speed) (time), we have $ds = |\alpha'(t)|\,dt$, so

$$\boxed{s(t) = (\textbf{arclength} \text{ from } \alpha(0) \text{ to } \alpha(t)\ ) = \int_0^t |\alpha'(u)|\,du.}$$

From this equation, we can take the derivative of both sides to get

$$\begin{array}{|l|}\hline \dot{s} = s'(t) = \dfrac{d}{dt}\displaystyle\int_0^t |\alpha'(u)|\,du \\ \hline \quad = |\alpha'(t)| = \textbf{speed} \\ \hline \end{array}$$

(the last part is from the Second Fundamental Theorem of Calculus: $\frac{d}{dx}\int_a^x f(u)du = f(x)$.).The **unit tangent vector** $T(t)$ is the unit vector in the direction of $\alpha'(t)$. That is, if the speed is non zero, then we can define $T(t)$ to be

$$\boxed{T(t) = \frac{1}{|\alpha'(t)|}\alpha'(t) = \frac{\dot{\alpha}}{\dot{s}} = \frac{1}{\dot{s}}(x'(t), y'(t), z'(t)).}$$

This is also called the **tangent vector field**, because it gives a vector at each point $\alpha(t)$ of the curve (i.e. all points, not just one point of the curve).

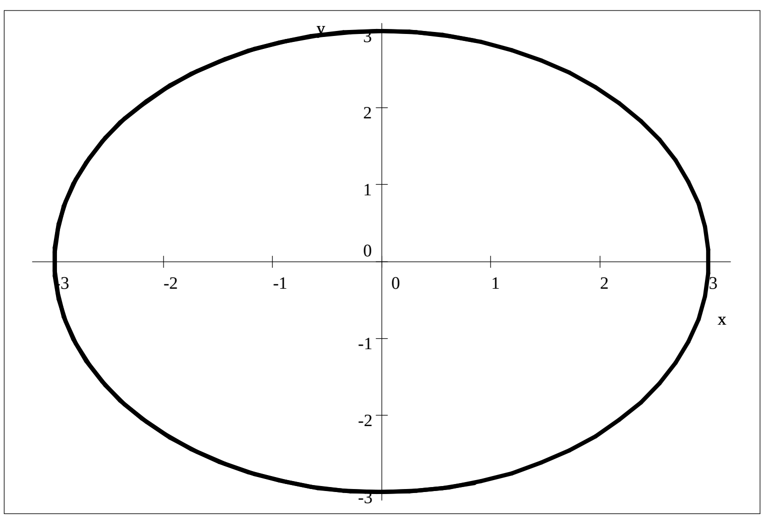
This calculation is a lot simpler if α is **unit speed**, ie $|\alpha'(t)| = \dot{s} = 1$. Then

$$\begin{array}{|c|}\hline \textbf{Unit speed curves}\ :\ T(t) = \alpha'(t) = (x'(t), y'(t), z'(t)), \\ \hline s(t) = t,\quad \dot{s} = 1. \\ \hline \end{array}$$

and $s(t) = t$. Another word for this: if $|\alpha'(t)| = \dot{s} = 1$, we say α is **parametrized by arclength**. A curve can always be reparametrized so that it is unit speed.

Examples:

Example *The circle of radius* 3*:* $\beta(t) = (3\cos(t), 3\sin(t))$.



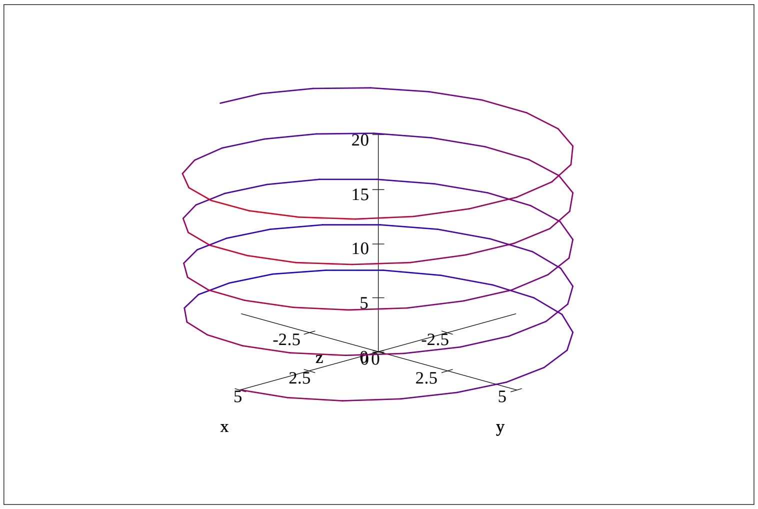
Velocity: $\beta'(t) = \left(\ -3\sin(t)\ ,\ 3\cos(t)\ \right)$.

Speed: $\dot{s} = |\beta'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = 3$.

Arclength: $s(t) = \int_0^t |\beta'(u)|\,du = \int_0^t 3\,du = 3t$.

Unit Tangent Vector: $T(t) = \frac{1}{\dot{s}}\beta'(t) = \frac{1}{3}\left(-3\sin(t)\ ,\ 3\cos(t) \right) = \left(-\sin(t)\ ,\ \cos(t) \right)$.

Example *A helix* $\eta(t) = (5\cos(3t), 5\sin(3t), 2t)$



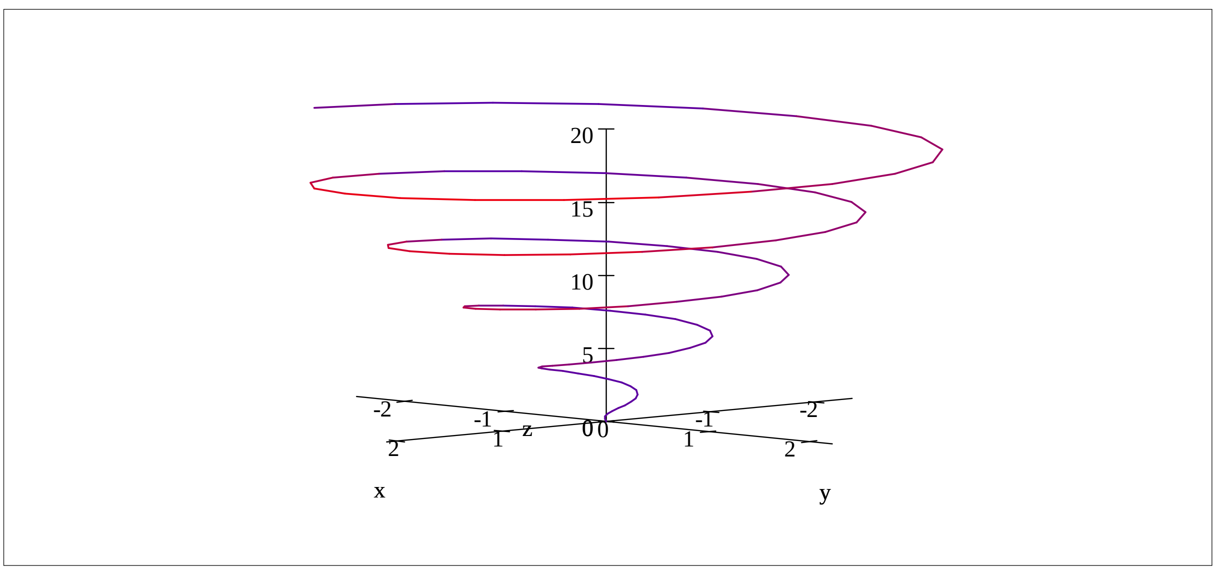
Velocity: $\eta'(t) = \left(-15\sin(3t)\ ,\ 15\cos(3t)\ ,\ 2 \right)$.

Speed: $\dot{s} = |\eta'(t)| = \sqrt{(-15\sin(3t))^2 + (15\cos(3t))^2 + 4} = \sqrt{229} \approx 15.132746$.

Arclength: $s(t) = \int_0^t |\eta'(u)|\,du = \int_0^t \sqrt{229}\ du = \sqrt{229}\,t$.

Unit Tangent Vector: $T(t) = \frac{1}{\dot{s}}\eta'(t) = \frac{1}{\sqrt{229}}\left(-15\sin(3t)\ ,\ 15\cos(3t)\ ,\ 2 \right) = \left(-\frac{15}{\sqrt{229}}\sin(3t)\ ,\ \frac{15}{\sqrt{229}}\cos(3t)\ ,\ \frac{2}{\sqrt{229}} \right)$.

Example *Inside Hallesan:* $\tilde{\eta}(t) = (5\frac{t}{20}\cos(3t), 5\frac{t}{20}\sin(3t), 2t)$



Velocity: $\tilde{\eta}'(t) = \left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\ ,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\ ,\ 2 \right)$.

Speed: $\dot{s} = |\tilde{\eta}'(t)| = \sqrt{\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\right)^2 + \left(\frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\right)^2 + 4} =$
$\sqrt{\frac{1}{16}(\cos^2(3t) + 9t^2\sin^2(3t) - 6t\cos(3t)\sin(3t)) + \frac{1}{16}(\sin^2(3t) + 9t^2\cos^2(3t) + 6t\cos(3t)\sin(3t)) + 4}$

$= \sqrt{\frac{1}{16}(1+9t^2) + 4} = \frac{1}{4}\sqrt{9t^2+65}.$

Arclength: $s(t) = \int_0^t |\tilde{\eta}'(u)|\, du = \int_0^t \frac{1}{4}\sqrt{9u^2+65}\, du =$
$\frac{1}{8}t\sqrt{9t^2+65} - \frac{65}{48}\ln 65 + \frac{65}{24}\ln\left(3t + \sqrt{9t^2+65}\right).$

Unit Tangent Vector:
$T(t) = \frac{1}{\dot{s}}\tilde{\eta}'(t) = \frac{1}{\frac{1}{4}\sqrt{9t^2+65}}\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2\right) =$
$= \left(\frac{1}{\sqrt{9t^2+65}}(\cos 3t - 3t\sin 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(\sin 3t + 3t\cos 3t)\,,\ \frac{8}{\sqrt{9t^2+65}}\right).$

Curvature and the Unit Normal Vector

The **curvature** $\kappa(t)$ is a measurement of the rate of turning of the curve α. Thus, curvature is the magnitude of the rate of change of the unit tangent vector with respect to arclength.

$$\textbf{curvature} = \left|\frac{d}{ds}T\right| = \left|\frac{\dot{T}}{\dot{s}}\right| = \left|\frac{T'(t)}{s'(t)}\right| = \left|\frac{1}{s'(t)}T'(t)\right|.$$

If curvature $\kappa(t) = 0$, then the curve is part of a straight line. (ie no turning)

To compute the curvature $\kappa(t)$, a scalar quantity, we must first calculate the vector $\frac{d}{ds}T = \frac{1}{s'(t)}T'(t)$, which we call the **curvature vector**. It gives more information that just $\kappa(t)$; it shows the *direction* of curving. We write

$$\text{curvature vector} = \frac{d}{ds}T = \frac{1}{s'(t)}T'(t) = \kappa(t)N(t)$$
$$= (\text{magnitude})(\text{unit vector}),$$

where $N(t)$ is the unit vector direction of turning. We call the vector $N(t)$ the **unit normal vector** or **unit normal vector field**. Thus

$$N(t) = \frac{1}{|T'(t)|}T'(t) = \frac{1}{s'(t)\kappa(t)}T'(t).$$

We can rearrange the equation to get:

$$\dot{T} = \dot{s}\kappa N. \qquad \text{\# (T derivative)}$$

An important thing to realize about N is that it is perpendicular to T. The reason for this is that since T is a unit vector, $T(t)\cdot T(t) = 1$. If we take the derivative of both sides of that equation, we get

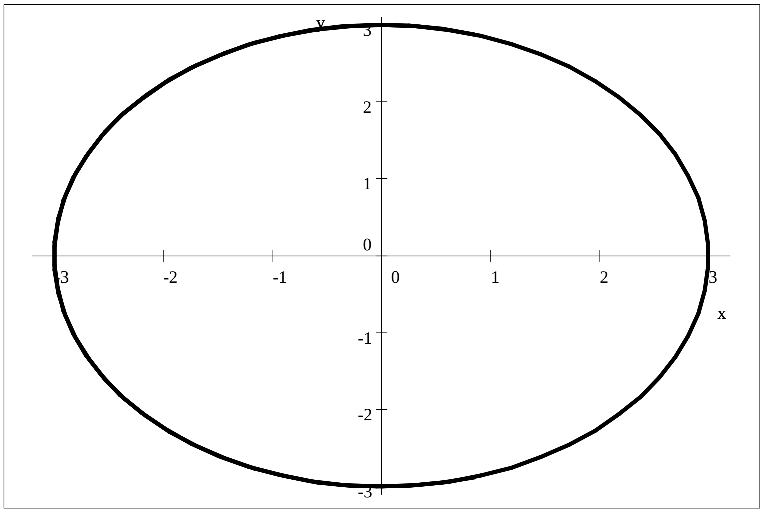
$$T'(t)\cdot T(t) + T(t)\cdot T'(t) = 0,$$

or $\dot{T}\cdot T = 0$. Thus $\dot{T}$ is perpendicular to T, and since $N = \frac{1}{\dot{s}\kappa}\dot{T}$ is a scalar function times $\dot{T}$, it also must be orthogonal to T. We have shown that T and N are two orthogonal unit vectors.

$$\boxed{T \cdot N = 0, \ \ |T| = 1, \ \ |N| = 1.}$$

Examples:

Example *The circle of radius* 3*:* $\beta(t) = (3\cos(t), 3\sin(t))$.



Velocity: $\beta'(t) = \left(-3\sin(t) \, , \, 3\cos(t) \right)$.

Speed: $\dot{s} = |\beta'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = 3$.

Arclength: $s(t) = \int_0^t |\beta'(u)| \, du = \int_0^t 3 \, du = 3t$.

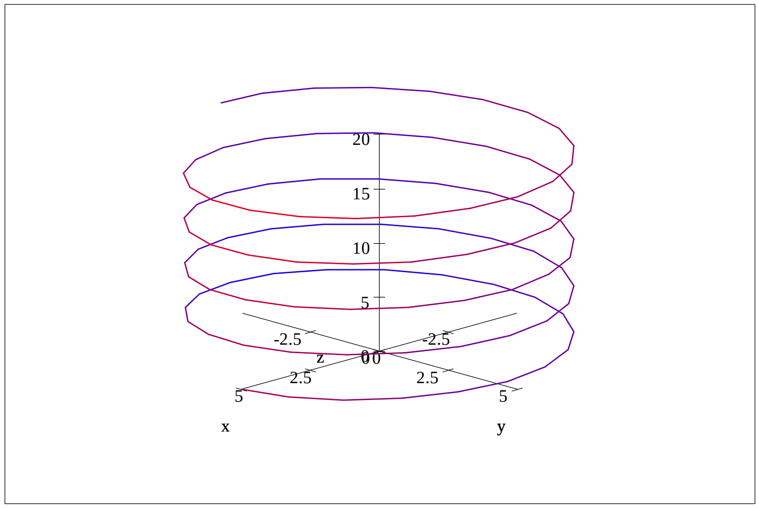
Unit Tangent Vector: $T(t) = \frac{1}{\dot{s}} \beta'(t) = \frac{1}{3} \left(-3\sin(t) \, , \, 3\cos(t) \right) = \left(-\sin(t) \, , \, \cos(t) \right)$.

Curvature vector:
$\frac{dT}{ds} = \frac{1}{\dot{s}} \dot{T} = \frac{1}{3} \left(-\cos(t) \, , \, -\sin(t) \right) = \left(-\frac{1}{3}\cos(t) \, , \, -\frac{1}{3}\sin(t) \right)$.

Curvature: $\kappa(t) = \left| \frac{dT}{ds} \right| = \left| \left(-\frac{1}{3}\cos(t) \, , \, -\frac{1}{3}\sin(t) \right) \right| = \frac{1}{3}$.

Unit Normal Vector: $N(t) = \frac{1}{\kappa} \frac{dT}{ds} = \left(-\cos(t) \, , \, -\sin(t) \right)$.

Example *A helix* $\eta(t) = (5\cos(3t), 5\sin(3t), 2t)$



Velocity: $\eta'(t) = \left(-15\sin(3t)\ ,\ 15\cos(3t)\ ,\ 2 \right)$.

Speed: $\dot{s} = |\eta'(t)| = \sqrt{(-15\sin(3t))^2 + (15\cos(3t))^2 + 4} = \sqrt{229} \approx 15.132746$.

Arclength: $s(t) = \int_0^t |\eta'(u)|\, du = \int_0^t \sqrt{229}\ du = \sqrt{229}\, t$.

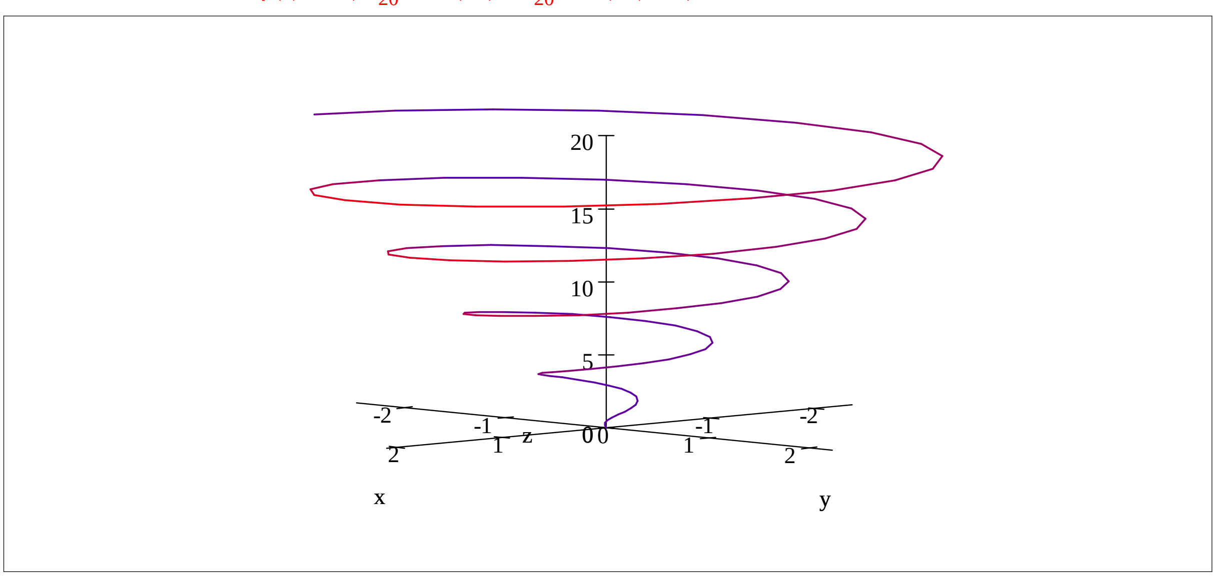
Unit Tangent Vector: $T(t) = \frac{1}{\dot{s}}\eta'(t) = \frac{1}{\sqrt{229}}\left(-15\sin(3t)\ ,\ 15\cos(3t)\ ,\ 2 \right) =$
$\left(-\frac{15}{\sqrt{229}}\sin(3t)\ ,\ \frac{15}{\sqrt{229}}\cos(3t)\ ,\ \frac{2}{\sqrt{229}} \right)$.

Curvature vector: $\frac{dT}{ds} = \frac{1}{\dot{s}}\dot{T} = \frac{1}{\sqrt{229}}\left(-\frac{15}{\sqrt{229}}3\cos(3t)\ ,-\frac{15}{\sqrt{229}}3\sin(3t)\ ,\ 0 \right)$
$= \left(-\frac{45}{229}\cos 3t\ ,\ -\frac{45}{229}\sin 3t\ ,\ 0 \right)$.

Curvature: $\kappa(t) = \left| \frac{dT}{ds} \right| = \left| \left(-\frac{45}{229}\cos 3t\ ,\ -\frac{45}{229}\sin 3t\ ,\ 0 \right) \right| = \frac{45}{229}$.

Unit Normal Vector: $N(t) = \frac{1}{\kappa}\frac{dT}{ds} = \left(-\cos 3t\ ,\ -\sin 3t\ ,\ 0 \right)$.

Example *Inside Hallesan:* $\tilde{\eta}(t) = (5\frac{t}{20}\cos(3t), 5\frac{t}{20}\sin(3t), 2t)$



Velocity: $\tilde{\eta}'(t) = \left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right)$.

Speed: $\dot{s} = |\tilde{\eta}'(t)| = \sqrt{(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t)^2 + (\frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t)^2 + 4} =$
$\sqrt{\frac{1}{16}(\cos^2(3t) + 9t^2\sin^2(3t) - 6t\cos(3t)\sin(3t)) + \frac{1}{16}(\sin^2(3t) + 9t^2\cos^2(3t) + 6t\cos(3t)\sin(3t)) + 4}$

$= \sqrt{\frac{1}{16}(1 + 9t^2) + 4} = \frac{1}{4}\sqrt{9t^2 + 65}$.

Arclength: $s(t) = \int_0^t |\tilde{\eta}'(u)|\, du = \int_0^t \frac{1}{4}\sqrt{9u^2 + 65}\ du =$
$\frac{1}{8}t\sqrt{9t^2 + 65} - \frac{65}{48}\ln 65 + \frac{65}{24}\ln\left(3t + \sqrt{9t^2 + 65}\right)$.

Unit Tangent Vector:
$T(t) = \frac{1}{\dot{s}}\tilde{\eta}'(t) = \frac{1}{\frac{1}{4}\sqrt{9t^2+65}}\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right) =$
$= \left(\frac{1}{\sqrt{9t^2+65}}(\cos 3t - 3t\sin 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(\sin 3t + 3t\cos 3t)\,,\ \frac{8}{\sqrt{9t^2+65}} \right)$.

Curvature vector:
$\frac{dT}{ds} = \frac{1}{\dot{s}}\dot{T} = \frac{1}{\frac{1}{4}\sqrt{9t^2+65}}\frac{d}{dt}\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right)$
$= \left(\frac{1}{\sqrt{9t^2+65}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(6\cos 3t - 9t\sin 3t)\,,\ 0 \right)$.

Curvature: $\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\left(\frac{1}{\sqrt{9t^2+65}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(6\cos 3t - 9t\sin 3t\right.\right.$

$= \frac{3}{\sqrt{9t^2+65}}\left|\left((-2\sin 3t - 3t\cos 3t)\,,\ (2\cos 3t - 3t\sin 3t)\,,\ 0 \right)\right|$

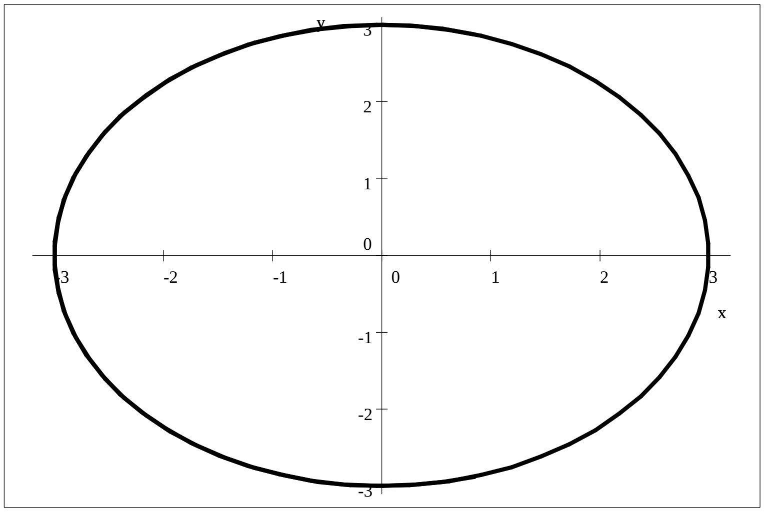
$= \frac{3}{\sqrt{9t^2+65}}\sqrt{(-2\sin 3t - 3t\cos 3t)^2 + (2\cos 3t - 3t\sin 3t)^2} = 3\frac{\sqrt{9t^2+4}}{\sqrt{9t^2+65}}$

Unit Normal Vector: $N(t) = \frac{1}{\kappa}\frac{dT}{ds}$

$= \left(\frac{1}{3\sqrt{9t^2+4}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{3\sqrt{9t^2+4}}(6\cos 3t - 9t\sin 3t)\,,\ 0 \right)$.

Another important intuitive interpretation of curvature is that the circle that fits the curve α best at a given point $\alpha(t)$ is a circle of radius $r(t) = \frac{1}{\kappa(t)}$ whose center is at $\alpha(t) + r(t)N(t)$. We call $r(t)$ the **radius of curvature**. This special circle is called the **osculating circle**. We can definitely see that makes sense in the first two examples above:

Example *The circle of radius* 3*:* $\beta(t) = (3\cos(t), 3\sin(t))$.

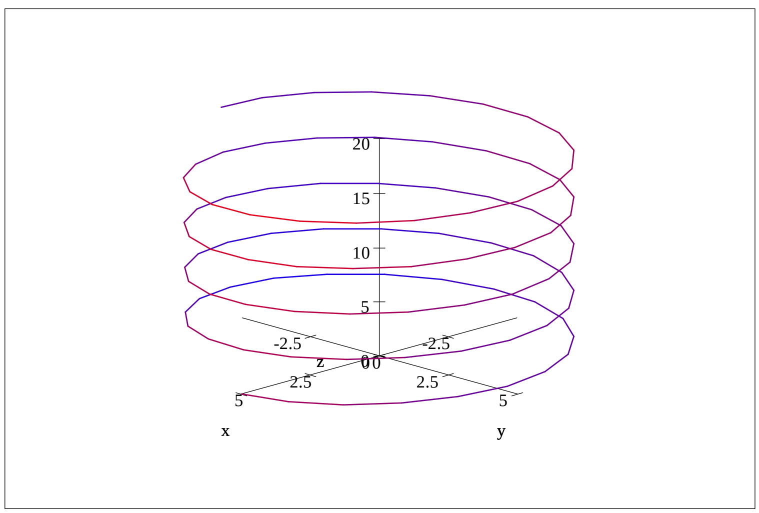


radius of curvature: $r(t) = \frac{1}{\kappa(t)} = \frac{1}{1/3} = 3$ *at each point.*

center of osculating circle: $\beta(t) + r(t)N(t)$

$= (3\cos(t), 3\sin(t)) + 3 \cdot \left(-\cos(t)\, ,\ -\sin(t) \right) = (0, 0).$

Example *A helix* $\eta(t) = (5\cos(3t), 5\sin(3t), 2t)$



radius of curvature: $r(t) = \frac{1}{\kappa(t)} = \frac{1}{\frac{45}{229}} = \frac{229}{45}$ *at each point.*

center of osculating circle: $\beta(t) + r(t)N(t)$

$= (5\cos(3t), 5\sin(3t), 2t) + \frac{229}{45} \cdot \left(-\cos 3t\, , -\sin 3t\, , 0 \right)$

$= (-\frac{4}{45}\cos(3t), -\frac{4}{45}\sin(3t), 2t).$

Binormal Vector and Torsion

Since T and N are two orthogonal unit vectors along α, it makes sense to define a third unit vector $B(t)$ that is orthogonal to both T and N. Let the **Unit Binormal Vector** be defined as

$$B = T \times N.$$

By the facts about the cross product, B is a unit vector orthogonal to both T and N, so that

$$B \cdot T = B \cdot N = 0, \; B \cdot B = 1.$$

We wish to see how this vector is related to the curve α. Start with the equation

$$N \cdot T = 0.$$

Differentiate both sides:

$$\dot{N} \cdot T + N \cdot \dot{T} = 0,$$

so that (from (ref: T derivative))

$$\begin{aligned} \dot{N} \cdot T &= -N \cdot \dot{T} = -N \cdot (\dot{s}\kappa N) \\ &= (-\dot{s}\kappa) N \cdot N = -\dot{s}\kappa \end{aligned} \qquad \text{\# (N' dot T formula)}$$

Next, start with

$$N \cdot N = 0.$$

Differentiate both sides:

$$\dot{N} \cdot N + N \cdot \dot{N} = 0,$$

so that

$$2\dot{N} \cdot N = 0. \qquad \text{\# (N' dot N formula)}$$

Thus, since (T, N, B) is a basis for R^3 (but centered at $\alpha(t)$), we must have

$$\dot{N} = (\#_1)T + (\#_2)N + (\#_3)B. \qquad \text{\# (N' equation)}$$

By equation (ref: N' dot T formula), if we dot both sides of equation (ref: N' equation) with T we get:

$$\begin{aligned} \dot{N} \cdot T &= (\#_1)T \cdot T + (\#_2)N \cdot T + (\#_3)B \cdot T \\ -\dot{s}\kappa &= (\#_1)1 + (\#_2)0 + (\#_3)0, \end{aligned}$$

so

$$(\#_1) = -\dot{s}\kappa.$$

Next, take the dot product of both sides of equation (ref: N' equation) with N and use (ref: N' dot N formula) we get:

$$\begin{aligned} \dot{N} \cdot N &= (\#_1)T \cdot N + (\#_2)N \cdot N + (\#_3)B \cdot N \\ 0 &= (\#_1)0 + (\#_2)1 + (\#_3)0, \end{aligned}$$

so

$$\boxed{(\#_2) = 0.}$$

So what about $(\#_3)$? We don't have any information about this, just that it is some number. We define this number to be the **torsion** $\tau(t)$ times speed $\dot{s}$ of the curve α at $\alpha(t)$. Thus,

$$\dot{N} = -\dot{s}\kappa T + \dot{s}\tau B. \qquad \# \text{ (N derivative)}$$

Thus a definition of torsion would be

$$\tau = \frac{1}{\dot{s}}(\dot{N} \cdot B).$$

The torsion measures the twisting of the curve out of the plane containing T and N. In fact, the curve α can be fit into a two-dimensional plane if and only if $\tau = 0$ at every point of the curve.

We can now also take the derivative of the binormal vector $B(t)$. Since $\{T, N, B\}$ is a basis of $\mathbb{R}^3$, there must be a formula like:

$$\dot{B} = (\#_1)T + (\#_2)N + (\#_3)B.$$

But then we now use the dot product formulas and the formula (ref: T derivative) for $\dot{T}$ and the formula (ref: N derivative) for $\dot{N}$:

$$\begin{array}{|c|}
\hline
B \cdot B = 1,\ B \cdot N = 0,\ B \cdot T = 0 \\
\hline
\text{derivative} \Rightarrow \dot{B} \cdot B + B \cdot \dot{B} = 2\dot{B} \cdot B = 0, \\
\hline
\dot{B} \cdot N + B \cdot \dot{N} = 0, \text{ so } \dot{B} \cdot N = -B \cdot \dot{N} = -\dot{s}\tau \\
\hline
\dot{B} \cdot T + B \cdot \dot{T} = 0, \text{ so } \dot{B} \cdot T = -B \cdot \dot{T} = 0 \\
\hline
\end{array}$$

Thus,

$$\begin{array}{l}
\dot{B} = 0T + (-\dot{s}\tau)N + 0B, \text{ i.e.} \\
\dot{B} = -\dot{s}\tau N
\end{array} \qquad \# \text{ (B derivative)}$$

We now put equations (ref: T derivative), (ref: N derivative), (ref: B derivative) into an array:

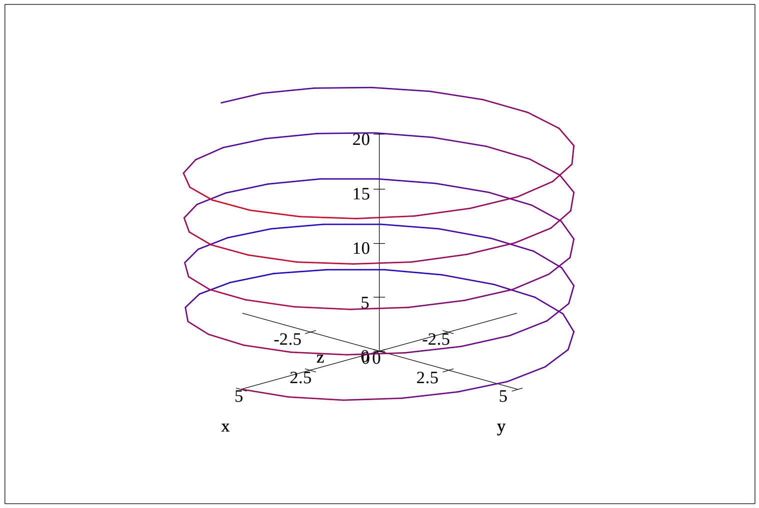
$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} \square & \dot{s}\kappa N & \square \\ -\dot{s}\kappa T & \square & +\dot{s}\tau B \\ \square & -\dot{s}\tau B & \square \end{pmatrix}, \text{ or}$$

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & +\dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

These formulas are called the **Frenet formulas**, and the set of data (κ, τ, T, N, B) is called the **Frenet apparatus**. The triple (T, N, B) is called the **Frenet frame**.

Examples:

Example *A helix* $\eta(t) = (5\cos(3t), 5\sin(3t), 2t)$



Velocity: $\eta'(t) = \big(-15\sin(3t)\,,\ 15\cos(3t)\,,\ 2\big)$.

Speed: $\dot{s} = |\eta'(t)| = \sqrt{(-15\sin(3t))^2 + (15\cos(3t))^2 + 4} = \sqrt{229} \approx 15.132746$.

Arclength: $s(t) = \int_0^t |\eta'(u)|\,du = \int_0^t \sqrt{229}\ du = \sqrt{229}\,t$.

Unit Tangent Vector: $T(t) = \frac{1}{\dot{s}}\eta'(t) = \frac{1}{\sqrt{229}}\big(-15\sin(3t)\,,\ 15\cos(3t)\,,\ 2\big) = \left(-\frac{15}{\sqrt{229}}\sin(3t)\,,\ \frac{15}{\sqrt{229}}\cos(3t)\,,\ \frac{2}{\sqrt{229}}\right)$.

Curvature vector: $\frac{dT}{ds} = \frac{1}{\dot{s}}\dot{T} = \frac{1}{\sqrt{229}}\left(-\frac{15}{\sqrt{229}}3\cos(3t)\,, -\frac{15}{\sqrt{229}}3\sin(3t)\,,\ 0\right)$
$= \left(-\frac{45}{229}\cos 3t\,,\ -\frac{45}{229}\sin 3t\,,\ 0\right)$.

Curvature: $\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\left(-\frac{45}{229}\cos 3t\,,\ -\frac{45}{229}\sin 3t\,,\ 0\right)\right| = \frac{45}{229}$.

Unit Normal Vector: $N(t) = \frac{1}{\kappa}\frac{dT}{ds} = \big(-\cos 3t\,,\ -\sin 3t\,,\ 0\big)$.

Unit Binormal Vector: $B = T \times N = \begin{vmatrix} i & j & k \\ -\frac{15}{\sqrt{229}}\sin 3t & \frac{15}{\sqrt{229}}\cos 3t & \frac{2}{\sqrt{229}} \\ -\cos 3t & -\sin 3t & 0 \end{vmatrix}$

$= \left(\frac{2\sin(3t)}{\sqrt{229}}\right)i - \left(\frac{2\cos(3t)}{\sqrt{229}}\right)j + \left(\frac{15}{\sqrt{229}}(\sin^2(3t) + \cos^2(3t))\right)k$

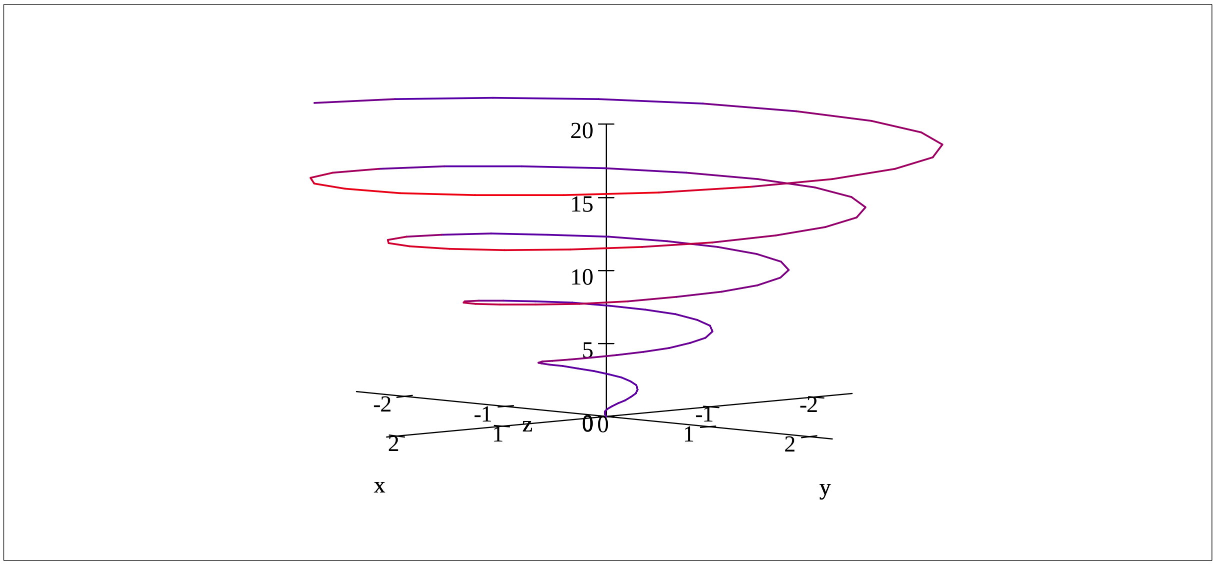
$= \left(\frac{2\sin(3t)}{\sqrt{229}}, -\frac{2\cos(3t)}{\sqrt{229}}, \frac{15}{\sqrt{229}}\right)$.

Torsion: $\dot{B} = -\dot{s}\tau N$, *so* $\dot{B} = \left(\frac{6\cos(3t)}{\sqrt{229}}, \frac{6\sin(3t)}{\sqrt{229}}, 0\right) = \sqrt{229}\,(\tau)\big(-\cos 3t\,, -\sin 3t\,, 0\big)$,

so $\tau = -\frac{6}{229}$.

Note that κ *and* τ *are both constant - this is always true for a circular helix.*

Example *Inside Hallesan:* $\tilde{\eta}(t) = (5\frac{t}{20}\cos(3t), 5\frac{t}{20}\sin(3t), 2t)$



Velocity: $\tilde{\eta}'(t) = \left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right).$

Speed: $\dot{s} = |\tilde{\eta}'(t)| = \sqrt{(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t)^2 + (\frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t)^2 + 4} =$
$\sqrt{\frac{1}{16}(\cos^2(3t) + 9t^2\sin^2(3t) - 6t\cos(3t)\sin(3t)) + \frac{1}{16}(\sin^2(3t) + 9t^2\cos^2(3t) + 6t\cos(3t)\sin(3t)) + 4}$

$= \sqrt{\frac{1}{16}(1 + 9t^2) + 4} = \frac{1}{4}\sqrt{9t^2 + 65}.$

Arclength: $s(t) = \int_0^t |\tilde{\eta}'(u)|\, du = \int_0^t \frac{1}{4}\sqrt{9u^2 + 65}\, du =$
$\frac{1}{8}t\sqrt{9t^2 + 65} - \frac{65}{48}\ln 65 + \frac{65}{24}\ln\left(3t + \sqrt{9t^2 + 65}\right).$

Unit Tangent Vector:
$T(t) = \frac{1}{\dot{s}}\tilde{\eta}'(t) = \frac{1}{\frac{1}{4}\sqrt{9t^2+65}}\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right) =$
$= \left(\frac{1}{\sqrt{9t^2+65}}(\cos 3t - 3t\sin 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(\sin 3t + 3t\cos 3t)\,,\ \frac{8}{\sqrt{9t^2+65}} \right).$

Curvature vector:
$\frac{dT}{ds} = \frac{1}{\dot{s}}\dot{T} = \frac{1}{\frac{1}{4}\sqrt{9t^2+65}}\frac{d}{dt}\left(\frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t\,,\ \frac{1}{4}\sin 3t + \frac{3}{4}t\cos 3t\,,\ 2 \right)$
$= \left(\frac{1}{\sqrt{9t^2+65}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(6\cos 3t - 9t\sin 3t)\,,\ 0 \right).$

Curvature: $\kappa(t) = \left|\frac{dT}{ds}\right| = \left|\left(\frac{1}{\sqrt{9t^2+65}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{\sqrt{9t^2+65}}(6\cos 3t - 9t\sin 3t\right.\right.$

$= \frac{3}{\sqrt{9t^2+65}}\left|\left((-2\sin 3t - 3t\cos 3t)\,,\ (2\cos 3t - 3t\sin 3t)\,,\ 0 \right)\right|$

$= \frac{3}{\sqrt{9t^2+65}}\sqrt{(-2\sin 3t - 3t\cos 3t)^2 + (2\cos 3t - 3t\sin 3t)^2} = 3\frac{\sqrt{9t^2+4}}{\sqrt{9t^2+65}}$

Unit Normal Vector: $N(t) = \frac{1}{\kappa}\frac{dT}{ds}$

$= \left(\frac{1}{3\sqrt{9t^2+4}}(-6\sin 3t - 9t\cos 3t)\,,\ \frac{1}{3\sqrt{9t^2+4}}(6\cos 3t - 9t\sin 3t)\,,\ 0 \right).$

Unit Binormal Vector:

$$B = T \times N = \left|\begin{array}{ccc} i & j & k \\ \frac{1}{\sqrt{9t^2+65}}(\cos 3t - 3t\sin 3t) & \frac{1}{\sqrt{9t^2+65}}(\sin 3t + 3t\cos 3t) & \frac{8}{\sqrt{9t^2+65}} \\ \frac{1}{\sqrt{9t^2+4}}(-2\sin 3t - 3t\cos 3t) & \frac{1}{\sqrt{9t^2+4}}(2\cos 3t - 3t\sin 3t) & 0 \end{array}\right|$$

$$= \left(\frac{8}{\sqrt{9t^2+65}}\frac{1}{\sqrt{9t^2+4}}(2\cos 3t - 3t\sin 3t)\right)i - \left(-\frac{8}{\sqrt{9t^2+65}}\frac{1}{\sqrt{9t^2+4}}(-2\sin 3t - 3t\cos 3t)\right)j$$

$$+\left(\frac{1}{\sqrt{9t^2+65}}(\cos 3t - 3t\sin 3t)\frac{1}{\sqrt{9t^2+4}}(2\cos 3t - 3t\sin 3t) - \frac{1}{\sqrt{9t^2+65}}(\sin 3t + 3t\cos 3t)\frac{1}{\sqrt{9t^2+4}}\right.$$

$$= \left(\frac{8}{\sqrt{9t^2+65}}\frac{1}{\sqrt{9t^2+4}}(2\cos 3t - 3t\sin 3t), -\frac{8}{\sqrt{9t^2+65}}\frac{1}{\sqrt{9t^2+4}}(-2\sin 3t - 3t\cos 3t), \frac{9t^2+2}{\sqrt{9t^2+4}\sqrt{9t^2+65}}\right.$$

Torsion: $\dot{N} = -\dot{s}\kappa T + \dot{s}\tau B$, *so* $\tau = \frac{1}{\dot{s}}\dot{N} \cdot B$. *Thus*

$$\dot{N} = \frac{d}{dt}\left(\frac{1}{3\sqrt{9t^2+4}}(-6\sin 3t - 9t\cos 3t), \frac{1}{3\sqrt{9t^2+4}}(6\cos 3t - 9t\sin 3t), 0\right)$$

$$\frac{d}{dt}\left(\frac{1}{3\sqrt{9t^2+4}}(-6\sin 3t - 9t\cos 3t)\right) =$$
$$\frac{1}{72t^2+81t^4+16}\left(54t(\sin 3t)\sqrt{9t^2+4} - 36(\cos 3t)\sqrt{9t^2+4} - 54t^2(\cos 3t)\sqrt{9t^2+4} + 81t^3(\sin 3t\right.$$

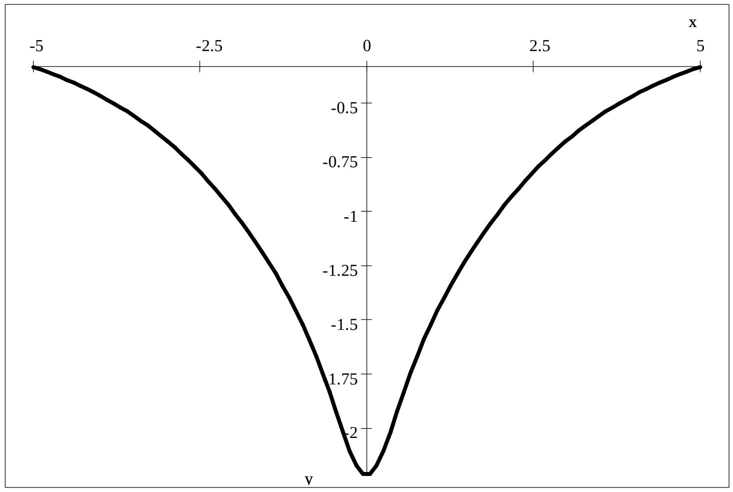
$$\frac{d}{dt}\left(\frac{1}{3\sqrt{9t^2+4}}(6\cos 3t - 9t\sin 3t)\right) =$$
$$\frac{1}{72t^2+81t^4+16}\left(-36(\sin 3t)\sqrt{9t^2+4} - 54t(\cos 3t)\sqrt{9t^2+4} - 81t^3(\cos 3t)\sqrt{9t^2+4} - 54t^2(\sin\right.$$

So $\tau = \frac{1}{\dot{s}}\dot{N} \cdot B$

$$= \frac{4}{\sqrt{9t^2+65}}\left(\begin{array}{l} \frac{1}{72t^2+81t^4+16}\left(54t(\sin 3t)\sqrt{9t^2+4} - 36(\cos 3t)\sqrt{9t^2+4} - 54t^2(\cos 3t)\sqrt{9t^2+4} + 81t^3(\right. \\ +\frac{1}{72t^2+81t^4+16}\left(-36(\sin 3t)\sqrt{9t^2+4} - 54t(\cos 3t)\sqrt{9t^2+4} - 81t^3(\cos 3t)\sqrt{9t^2+4} - 54t^2\right. \end{array}\right.$$

$$\tau = \frac{-288(3t^2+2)}{(9t^2+65)(9t^2+4)}$$

Torsion τ *as a function of time* t *:*



Fundamental Theorem of Space Curves

One very important fact about curves in $\mathbb{R}^3$ is that the curvature and torsion completely determine the geometry of the curve. That is, two directed curves are congruent if and only if when parametrized by arclength (ie unit speed) in the correct direction, the curvature and torsion functions are the same.

Why does that matrix look like that?

You may have noticed that the matrix in the Frenet equation is skew-symmetric ($M^T = -M$):

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & +\dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

There is a very good Linear Algebra reason for this. First of all, we actually have a 3×3 matrix equation above, if you write out the components of the Frenet frame:

$$\frac{d}{dt}\begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix} = \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & \dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix}\begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}.$$

The next thing to notice is that since (T, N, B) is an orthonormal set of three vectors in $\mathbb{R}^3$, the matrix $F = \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$ must be an **orthogonal matrix**, meaning that $FF^T = I$. You can check this:

$$FF^T = \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}^T = \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} T_1 & N_1 & B_1 \\ T_2 & N_2 & B_2 \\ T_3 & N_3 & B_3 \end{pmatrix}$$

$$= \begin{pmatrix} T \cdot T & T \cdot N & T \cdot B \\ N \cdot T & N \cdot N & N \cdot B \\ B \cdot T & B \cdot N & B \cdot B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Another way to say this is that $F^{-1} = F^T$ and $(F^T)^{-1} = F$. If you take the derivative of both sides of the equation $FF^T = I$, you get

$$\dot{F}F^T + F\dot{F}^T = 0, \text{ or}$$
$$\dot{F}F^T = -F\dot{F}^T$$
$$= -(\dot{F}F^T)^T$$

since $(AB)^T = B^T A^T$. This tells you that if F is an orthogonal matrix depending on t, then the matrix $M = \dot{F}F^T$ satisfies $M = -M^T$; that is, M is skew-symmetric. If we plug in our special orthogonal matrix, we get

$$M = \left(\frac{d}{dt}F\right)F^T = \left(\frac{d}{dt}\begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}\right)\begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}^T$$

$$= \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & \dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix} \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix}^T$$

$$= \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & \dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix}.$$

So the fact it makes sense that $M = \begin{pmatrix} \square & \dot{s}\kappa & \square \\ -\dot{s}\kappa & \square & \dot{s}\tau \\ \square & -\dot{s}\tau & \square \end{pmatrix}$ is skew-symmetric because of Linear Algebra. By the way, what we have essentially shown here is that "the Lie algebra of $O(3)$ is the set of skew-symmetric 3×3 matrices". If you will go further in mathematics, you will understand what that means.