Series

Example of a series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$\sqrt[10]{e} \approx 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}$$
$$= 1.1051666666\ldots$$
$$(\text{actual value} \;:\; \sqrt[10]{e} = 1.105170918\ldots)$$

Others:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$= 1 + x + x^2 + x^3 + \ldots$$

Now, for something completely different

The Heat Equation on the real line:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) K(t,x,y) = 0$$
$$\lim_{t\to 0} K(t,x,y) = \delta(x-y), \quad \text{which means}$$
$$\lim_{t\to 0} \int_{\phi=0}^{2\pi} K(t,x,y) f(y)\, dy = f(x) \text{ for any smooth function } f$$

This K is called the *heat kernel*, or the *fundamental solution of the heat equation*.

Initial Value Problem (IVP) for the Heat Equation: find $u(t,x)$, the temperature at time t at position x, given the initial temperature distribution $f(x)$.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = 0$$
$$u(0,x) = f(x).$$

Answer:

$$u(t,x) = \int_{\phi=0}^{2\pi} K(t,x,y) f(y)\, dy.$$

(check it - it works!)

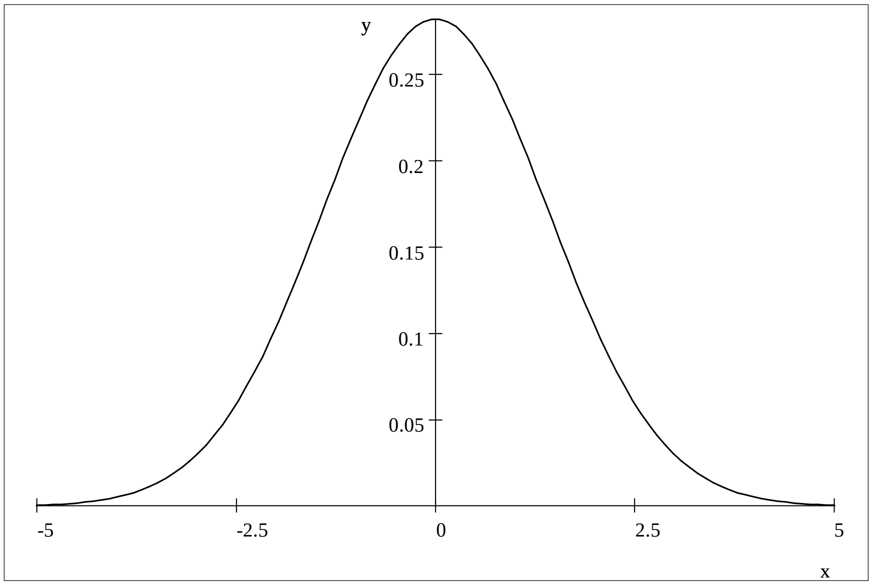
By the way,

$$K(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)}.$$

Check: $\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/(4t)}\right)$

$y = \frac{1}{\sqrt{4\pi t}} e^{-(x)^2/(4t)}$

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi 17}} e^{-x^2/(4*17)} dx$



You can do this on any curve, surface, manifold. Examples: half the real line, pretend you have insulated the boundary (ie $\frac{\partial}{\partial x} u(t,0) = 0$), the circle.... How about the circle?

The **Heat Equation** on the circle:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \theta^2}\right) K(t,\theta,\phi) = 0$$

$$\lim_{t \to 0} K(t,\theta,\phi) = \delta_{\theta\phi}, \quad \text{which means}$$

$$\lim_{t \to 0} \int_{\phi=0}^{2\pi} K(t,\theta,\phi) f(\phi)\, d\phi = f(\theta) \text{ for any smooth function } f$$

Initial Value Problem (IVP) for the Heat Equation: find $u(t,\theta)$, the temperature at time t at position θ, given the initial temperature distribution $f(\theta)$.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial \theta^2}\right) u(t,\theta) = 0$$

$$u(0,\theta) = f(\theta).$$

Answer:

$$u(t,\theta) = \int_{\phi=0}^{2\pi} K(t,\theta,\phi) f(\phi)\, d\phi.$$

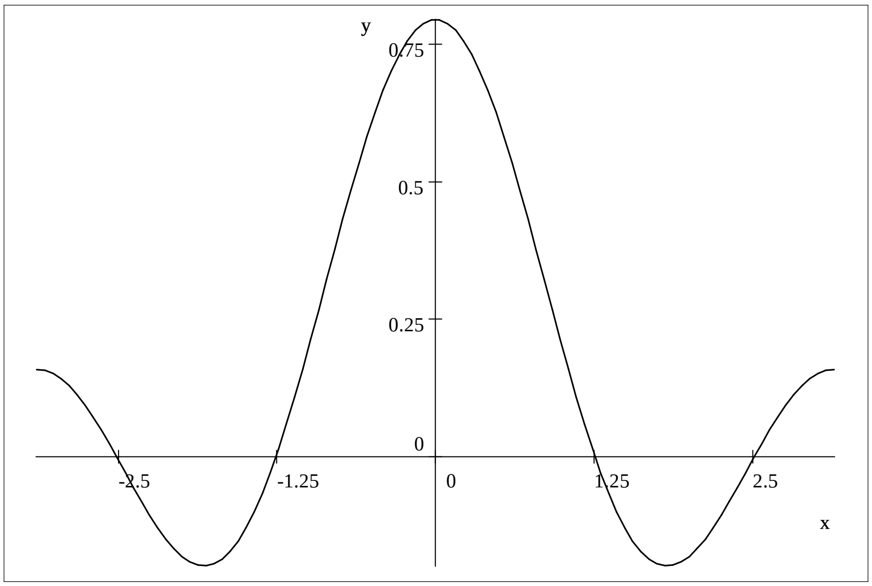
(check it - it works!)

One thing: What is $K(t,\theta,\phi)$???????

$$K(t,\theta,\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(n(\theta-\phi))$$
$$= \frac{1}{2\pi} + \frac{1}{\pi}(e^{-t}\cos(\theta-\phi) + e^{-4t}\cos(2(\theta-\phi)) + \dots)$$

Check: $\frac{\partial}{\partial t}\left(e^{-n^2 t}\cos(n(\theta-\phi))\right)$
$-\frac{\partial^2}{\partial\theta^2}\left(e^{-n^2 t}\cos(n(\theta-\phi))\right)$

$\int_{-\pi}^{\pi}\left[\frac{1}{2\pi} + \frac{1}{\pi}\left(\sum_{n=1}^{9} e^{-n^2 t}\cos(n\theta)\right)\right]d\theta$
$= 1 + \frac{1}{\pi}\sum_{n=1}^{9} e^{-n^2 t}\int_{-\pi}^{\pi}\cos(n\theta)d\theta$
$y = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{2} e^{-n^2 0.0005}\cos(nx)$



But wait — here's a totally different way to get the heat kernel on the circle: Imagine the real line (except with θ as the variable), and at $t = 0$, a certain amount of heat is applied at each $\theta = 2k\pi$ for $k \in \mathbb{Z}$. The resulting temperature at position θ at time t would be

$$\dots + \frac{1}{\sqrt{4\pi t}}e^{-(\theta+2\pi)^2/(4t)} + \frac{1}{\sqrt{4\pi t}}e^{-(\theta-0)^2/(4t)} + \frac{1}{\sqrt{4\pi t}}e^{-(\theta-2\pi)^2/(4t)} + \frac{1}{\sqrt{4\pi t}}e^{-(\theta-4\pi)^2/(4t)} + \frac{1}{\sqrt{4\pi t}}e^{-(\theta-6\pi)^2/(4t)} + \dots$$
$$= \sum_{k\in\mathbb{Z}}\frac{1}{\sqrt{4\pi t}}e^{-(\theta-2k\pi)^2/(4t)}$$

But this must be the same as

$$K(t,\theta,0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(n(\theta-0)), \text{ so}$$
$$K(t,\theta,0) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(n\theta) = \frac{1}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}} e^{-(\theta-2k\pi)^2/(4t)}$$

In greater generality, where heat is applied at $\phi + 2k\pi$ for $k \in \mathbb{Z}$, we get

$$K(t,\theta,\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty} e^{-n^2 t}\cos(n(\theta-\phi)) = \frac{1}{\sqrt{4\pi t}}\sum_{k\in Z} e^{-(\theta-\phi-2k\pi)^2/(4t)}.$$

Now, plug in $\theta = \phi$:

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t} = \frac{1}{\sqrt{4\pi t}}\sum_{k\in Z} e^{-k^2\pi^2/t}, \text{ or}$$

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t} = \frac{1}{\sqrt{4\pi t}} + \frac{2}{\sqrt{4\pi t}}\sum_{k=1}^{\infty} e^{-k^2\pi^2/t}$$

If $t = \pi$, this is kind of boring, but for example if $t = 1$ this says

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2} = \frac{1}{\sqrt{4\pi}}\sum_{k\in Z} e^{-k^2\pi^2} = \frac{1}{\sqrt{4\pi}} + \frac{2}{\sqrt{4\pi}}\sum_{k=1}^{\infty} e^{-k^2\pi^2}, \text{ or}$$

$$\frac{1}{2\pi}\left(1 + \frac{2}{e} + \frac{2}{e^4} + \frac{2}{e^9} + \ldots\right) = \frac{1}{\sqrt{4\pi}}\left(1 + \frac{2}{e^{\pi^2}} + \frac{2}{e^{4\pi^2}} + \frac{2}{e^{9\pi^2}} + \ldots\right)$$

Note:

$\frac{1}{2\pi}(1) \approx 0.1591549431$

$\frac{1}{2\pi}(1 + \frac{2}{e}) \approx 0.2762546061$

$\frac{1}{2\pi}\left(1 + \frac{2}{e} + \frac{2}{e^4}\right) \approx 0.2820846551$

$\frac{1}{2\pi}\left(1 + \frac{2}{e} + \frac{2}{e^4} + \frac{2}{e^9}\right) \approx 0.2821239376$

$\frac{1}{\sqrt{4\pi}}(1) \approx 0.2820947918$

$\frac{1}{\sqrt{4\pi}}\left(1 + \frac{2}{e^{\pi^2}}\right) \approx 0.2821239735$

$\frac{1}{\sqrt{4\pi}}\left(1 + \frac{2}{e^{\pi^2}} + \frac{2}{e^{4\pi^2}}\right) \approx 0.2821239735$

$\frac{1}{\sqrt{4\pi}}\left(1 + \frac{2}{e^{\pi^2}} + \frac{2}{e^{4\pi^2}} + \frac{2}{e^{9\pi^2}}\right) \approx 0.2821239735$

Note the difference in rates of convergence!