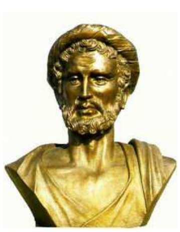
Mathematics and Music

What?

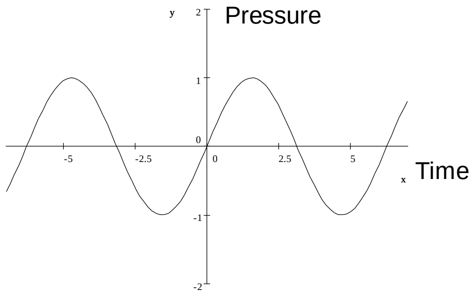


Archytas,
Pythagoras
Other Pythagorean
Philosophers/Educators:

The Quadrivium

- **Mathematics** (“study of the unchangeable”)
  - Number
    - Arithmetic *numbers at rest*
    - Music *numbers in motion*
  - Magnitude
    - Geometry *magnitudes at rest*
    - Astronomy *magnitudes in motion*

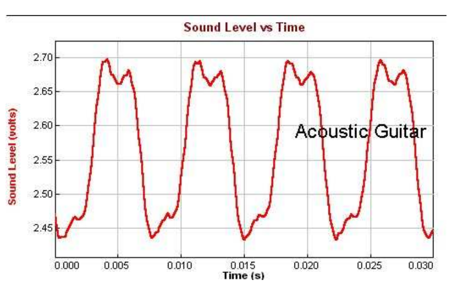
Physics of Sound and Musical Tone

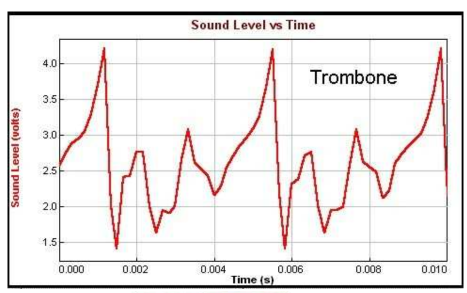


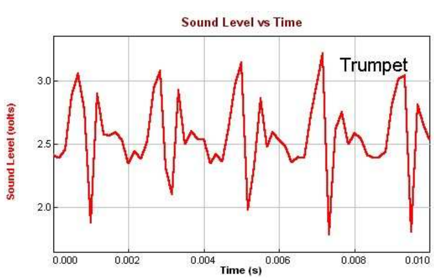
Pitch: frequency of wave = number of cycles per second (Hz)
higher pitch ⇔ more cycles per second ⇔ skinnier waves on graph

Volume: amplitude of wave = difference between maximum pressure and average pressure
higher volume ⇔ taller waves on graph

Timbre: quality of tone = shape of wave







Musical Scale:
(increasing pitch = frequency)
C→D→E→F→G→A→B→C→…

Notes between: **$C^{\#}$ $D^{\#}$ $F^{\#}$ $G^{\#}$ $A^{\#}$**
or $D^{\flat}$ $E^{\flat}$ $G^{\flat}$ $A^{\flat}$ $B^{\flat}$

12-note Musical Scale (Chromatic Scale):
C→ $C^{\#}$→ D→ $E^{\flat}$→ E→ F→ $F^{\#}$→ G→ $G^{\#}$→A→$B^{\flat}$→B→C→…

Musical Intervals:

Interval Name		Examples
2^{nd}	2 half steps	**C→D, E→ $F^{\#}$**
3^{rd}	4 half steps	**G→B, B→ $D^{\#}$**
5^{th}	7 half steps	**G→D, B→ $F^{\#}$**
Octave	12 half steps	**C_1→C_2, $F^{\#}_2$→ $F^{\#}_3$**

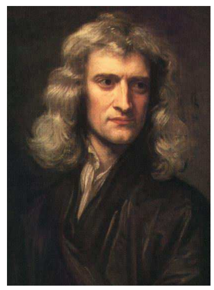
<u>Harmonics</u> (Partials)

Multiply Frequency by	Interval Produced	Example C_1=fundamental
2	1 octave	C_2
3	1 octave + perfect 5th (2% sharp)	G_2
4	2 octaves	C_3
5	2 octaves + major 3rd (14% flat)	E_3
6	2 octaves + perfect 5th (2% sharp)	G_3
7	2 octaves + dominant 7th (32% flat)	$B^{\flat}_3$
8	3 octaves	C_4
9	3 octaves + whole step (4% sharp)	D_4
10	3 octaves + major 3rd (14% flat)	E_4

When a musical instrument is played, the harmonics appear at different amplitudes --- this creates the different timbres.

A branch of mathematics – *Fourier analysis* – deals with decomposing a wave of a certain frequency into its harmonic components.
Joseph Fourier (1768 – 1830) discovered these methods and utilized them to solve heat flow problems.

All of this mathematics uses Calculus in an essential way (discovered by Newton and Leibniz independently in late 1600s).



Creating new musical tones using harmonics from just one musical tone:

Given tone: C_3 (tuned to frequency **262 Hz**)
How can we make E_3?

Let me count the ways:

Method 1: "Just" tuning
- Multiply frequency by 5 $C_3 \to E_5$ (1310 Hz)
- Multiply frequency by $\frac{1}{4}$ $E_5 \to E_3$ (**327.5 Hz**)
- Corresponds to factor of $\frac{5}{4}$ for Major 3rd interval

Method 2: "Pythagorean" tuning
- Multiply frequency by 3 $C_3 \to G_4$ (786 Hz)
- Multiply frequency by $\frac{1}{2}$ $G_4 \to G_3$ (393 Hz)
- Multiply frequency by 3 $G_3 \to D_5$ (1179 Hz)
- Multiply frequency by $\frac{1}{4}$ $D_5 \to D_3$ (294.75 Hz)
- Multiply frequency by 3 $D_3 \to A_4$ (884.25 Hz)
- Multiply frequency by $\frac{1}{2}$ $A_4 \to A_3$ (442.125 Hz)
- Multiply frequency by 3 $A_3 \to E_5$ (1326.375 Hz)
- Multiply frequency by $\frac{1}{4}$ $E_5 \to E_3$ (**331.6 Hz**)

- Corresponds to factor of $\frac{3^4}{2^6}$ for Major 3rd interval

Yet another example: the "Pythagorean comma"

The cycle of "perfect" fifths:

Starting tone: C_1 (tuned to frequency **65 Hz**)

- Multiply frequency by 3 $C_1 \rightarrow$ G (195 Hz)
- Multiply frequency by $\frac{1}{2}$ $G_2 \rightarrow G_1$ (97.5 Hz)
- Corresponds to factor of $\frac{3}{2} = 1.5$ for Perfect 5th interval

Keep doing that:

$G_1 \rightarrow D_2$ (146.25 Hz) $\rightarrow A_2$ (219.38 Hz)
$\rightarrow E_3$ (329.06 Hz) $\rightarrow B_3$ (493.59 Hz)
$\rightarrow F^{\#}_4$ (740.39 Hz) $\rightarrow C^{\#}_5$ (1110.6 Hz)
$\rightarrow G^{\#}_5$ (1665.9 Hz) $\rightarrow E^{\flat}_6$ (2498.8 Hz)
$\rightarrow B^{\flat}_6$ (3748.2 Hz) $\rightarrow F_7$ (5622.3 Hz)
$\rightarrow C_8$ (8433.5 Hz) $\rightarrow$ (go down 7 octaves)
$\rightarrow C_1$ (**65.8868 Hz**)
Wait a minute!!!!

The frequency (pitch) is high by a factor of

$$\frac{3^{12}}{2^{19}} \approx 1.013643....$$

(the *Pythagorean comma*).

Evenly-spaced intervals between octaves: Equal tempering

Pythagorean tuning *(popular through 16th century)*:

C to G is a perfect fifth – factor of $\frac{3}{2} = 1.5$

F# to C# is a perfect fifth – factor of $\frac{2^{18}}{3^{11}} = \frac{1.5}{comma} \approx 1.48$

Thus music with C's and G's sounds good.
Music with F#'s and C#'s sounds a little weird.

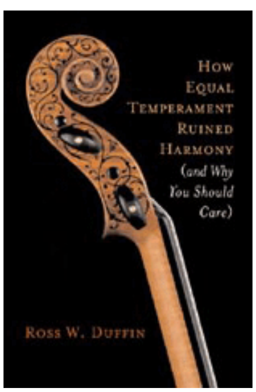
Equal-tempered tuning *(introduced by Simon Stevin (mathematician) in 1596; in 1630s Father Mersenne formulated rules for tuning by beats; became popular in 18th century)*:
All intervals are the same in all keys.
All keys sound roughly the same.

A half-step interval is a factor of $\sqrt[12]{2} = 2^{1/12}$

Thus, there are 12 even half-steps between octaves.

Interval	Example	Frequency Multiple (Pythagorean)	Frequency Multiple (Even-tempered)
half step	C - C#	$\frac{2^8}{3^5} = \frac{256}{243} \approx 1.053$	$2^{1/12} \approx 1.059$
whole step	C - D	$\frac{3^2}{2^3} = \frac{9}{8} = 1.125$	$2^{2/12} \approx 1.122$
minor third	C - E	$\frac{2^5}{3^3} = \frac{32}{27} \approx 1.185$	$2^{3/12} \approx 1.189$
major third	C - E	$\frac{3^4}{2^6} = \frac{81}{64} \approx 1.266$	$2^{4/12} \approx 1.260$
perfect fourth	C - F	$\frac{4}{3} \approx 1.333$	$2^{5/12} \approx 1.335$
perfect fifth	C - G	$\frac{3}{2} = 1.5$	$2^{7/12} \approx 1.498$
octave	C_1 - C_2	2	$2^{12/12} = 2$

From W. W. Norton Catalog:



A captivating look at how musical temperament evolved, and how we could (and perhaps should) be tuning differently today.

Ross W. Duffin presents an engaging and elegantly reasoned exposé of musical temperament and its impact on the way in which we experience music. A historical narrative, a music theory lesson, and, above all, an impassioned letter to musicians and listeners everywhere, *How Equal Temperament Ruined Harmony* possesses the power to redefine the very nature of our interactions with music today.

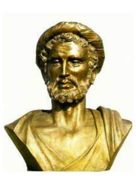
For nearly a century, equal temperament—the practice of dividing an octave into twelve equally proportioned half-steps—has held a virtual monopoly on the way in which instruments are tuned and played. In his new book, Duffin explains how we came to rely exclusively on equal temperament by charting the fascinating evolution of tuning through the ages. Along the way, he challenges the widely held belief that equal temperament is a perfect, "naturally selected" musical system, and proposes a radical reevaluation of how we play and hear music. **Ross W. Duffin**, author of *Shakespeare's Songbook* (winner of the Claude V. Palisca Award), is Fynette H. Kulas Professor of Music at Case Western Reserve University. He lives in Shaker Heights, Ohio.

Pairs of Harmonious Tones: tones sound harmonious when they are played together if they share common harmonics with nearly the same frequencies.

Octave Interval	Lower Pitch Harmonic Frequencies	Higher Pitch Harmonic Frequencies
	f	$2f$
	$2f$	$4f$
	$3f$	$6f$
	$4f$	$8f$
	$5f$	$10f$
	$6f$	$12f$
	…	…

Pairs of Harmonious Tones

Pythagorean Perfect 4th	Lower Pitch Harmonic Frequencies	Higher Pitch Harmonic Frequencies
	f	$\frac{4}{3}f$
	$2f$	$\frac{8}{3}f$
	$3f$	$4f$
	$4f$	$\frac{16}{3}f$
	$5f$	$\frac{20}{3}f$
	$6f$	$8f$
	…	…



Pairs of Harmonious Tones

Just Major 3rd	**Lower Pitch Harmonic Frequencies**	**Higher Pitch Harmonic Frequencies**
	f	$\frac{5}{4}f$
	$2f$	$\frac{10}{4}f$
	$3f$	$\frac{15}{4}f$
	$4f$	$5f$
	$5f$	$\frac{25}{4}f$
	$6f$	$\frac{30}{4}f$
	…	…

The intervals in harmonious order (mathematically determined)

Interval	Rational approx.	denominator
octave	2/1	1
Perfect 5th	3/2	2
Perfect 4th	4/3	3
Major 6th	5/3	3
Major 3rd	5/4	4
Minor 3rd	6/5	5
Tritone	7/5	5
Augmented 5th	8/5	5
Minor 7th	9/5	5
Major 2nd	9/8	8
Major 7th	15/8	8
Minor 2nd	16/15	15

Note: Hindemith 1930s: *The Craft of Musical Composition*

The 3-note chords in harmonious order (consonant to dissonant)

Chord	ratios	LCD, sum of degrees
C-F-A	4/3,5/3,5/4	LCD=3, sum=10
C-E-G	5/4,3/2,6/5	LCD=4, sum=11
C-E♭-A♭	6/5,8/5,4/3	LCD=5, sum=13
C-F-G	4/3,3/2,9/8	LCD=6, sum=13
C-E♭-G	6/5,3/2,5/4	LCD=10, sum=11
….	….	….
….	….	….
C-C#-B	17/16,15/8,15/34	LCD=16, sum=58
C-C#-D	16/15,9/8,135/128	LCD=120, sum=151

Would Space Aliens want to listen to Mozart?

To construct an even-tempered scale that includes the first nontrivial harmonic, we need to find a fraction a/b such that

$$3/2 \approx 2^{a/b}$$

The resulting chromatic scale would have b distinct notes. We would have

$$\frac{a}{b} \approx \log_2 (3/2) = .5849625007\ 21...$$

Possible *a/b*	Decimal	Error in 5th
3/5	*.6*	1.05%
4/7	*.5714*	0.93%
7/12	*.5833*	0.11%
17/29	*.5862*	0.086%
65/111	*.5856*	0.043%