THE SPECTRUM OF BASIC DIRAC OPERATORS

KEN RICHARDSON

Abstract. In this note, we discuss Riemannian foliations, which are smooth foliations that have a transverse geometric structure. We explain a known generalization of Dirac-type operators to transverse operators called "basic Dirac operators" on Riemannian foliations, which require the additional structure of what is called a bundle-like metric. We explain the result in [10] that the spectrum of such an operator is independent of the choice of bundle-like metric, provided that the transverse geometric structure is fixed. We discuss consequences, which include defining a new version of the exterior derivative and de Rham cohomology that are nicely adapted to this transverse geometric setting.

1. Introduction

The content here concerns some work in [10] and also briefly mentions work in [21], [16], and [3]; it also provides applications not given in the these references.

1.1. **Smooth foliations and basic forms.** Let $(M, \mathcal{F})$ be a smooth, closed manifold of dimension n endowed with a foliation $\mathcal{F}$ given by an integrable subbundle $L \subset TM$ of rank p. The set $\mathcal{F}$ is a partition of M into immersed submanifolds (*leaves*) such that the transition functions for the local product neighborhoods (foliation charts) are smooth. The subbundle $L = T\mathcal{F}$ is the tangent bundle to the foliation; at each $p \in M$, $T_p\mathcal{F} = L_p$ is the tangent space to the leaf through p.

Many researchers have studied basic forms and basic cohomology, especially in the particular cases of Riemannian foliations with bundle-like metrics, to be discussed later (see [1], [14], [28]). Basic forms are differential forms on M that locally depend only on the transverse variables in the foliation charts — that is, forms α satisfying $X \lrcorner \alpha = X \lrcorner d\alpha = 0$ for all $X \in \Gamma(L)$; the symbol "$\lrcorner$" stands for interior product. Let $\Omega(M, \mathcal{F}) \subset \Omega(M)$ denote the space of basic forms. These differential forms are preserved by the exterior derivative and are used to define basic cohomology groups, which can be infinite-dimensional but are always finite-dimensional in the case of Riemannian foliations. We define the *basic cohomology group* $H^k(M, \mathcal{F})$ by

$$H^k(M, \mathcal{F}) = \frac{\ker d_k}{\mathrm{Im} d_{k-1}}$$

with

$$d_k = d : \Omega^k(M, \mathcal{F}) \to \Omega^{k+1}(M, \mathcal{F}).$$

We comment that the point of using basic forms is an effort to find a form of de Rham cohomology on a singular, possibly non-Hausdorff space, that space being the set of leaves of the foliation. To gain the smooth structure, we loose a bit of information about the leaf space. The basic cohomology can be infinite-dimensional, and it can be relatively trivial.

2000 *Mathematics Subject Classification.* 53C12; 53C21; 58J50; 58J60.

Key words and phrases. Riemannian foliation, Dirac operator, transverse geometry, eigenvalue estimate, basic cohomology.

We may also define basic cohomology with values in a foliated vector bundle; by doing this we gain more topological information about the leaf space.

Basic cohomology does not necessarily satisfy Poincaré duality, even if the foliation is transversally oriented. If there are additional restrictions, it may satisfy duality, for example if the manifold admits a metric for which the leaves are locally equidistant and are minimal submanifolds (ie a taut Riemannian foliation). We emphasize that basic cohomology is a smooth foliation invariant and does not depend on the choice of metric or any transverse or leafwise structure.

1.2. **Riemannian foliations and bundle-like metrics.** We assume throughout the paper that the foliation is *Riemannian*; this means that there is a metric on the local space of leaves — a holonomy-invariant transverse metric g_Q on the normal bundle $Q = TM \diagup L$; this means that the transverse Lie derivative $\mathcal{L}_X g_Q$ is zero for all leafwise vector fields $X \in \Gamma(L)$. This metric is a substitute for a metric on the singular space of leaves. This condition is characterized by the existence of a unique metric and torsion-free connection ∇ on Q [20], [24], [28]. We can then associate to ∇ the transversal curvature data, in particular the transversal Ricci curvature Ric^{∇} and transversal scalar curvature Scal^{∇}.

We often assume that the manifold is endowed with the additional structure of a *bundle-like metric* [24], i.e. the metric g on M induces the metric on $Q \simeq L^{\perp}$. Every Riemannian foliation admits bundle-like metrics that are compatible with a given $(M, \mathcal{F}, g_Q)$ structure. There are many choices, since one may freely choose the metric along the leaves and also the transverse subbundle $N\mathcal{F}$. We note that a bundle-like metric on a smooth foliation is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant.

There are topological restrictions to the existence of bundle-like metrics (and thus Riemannian foliations). Important examples of requirements for the existence of a Riemannian foliations include

- certain characteristic classes must vanish (see [15])
- leaf closures must partition the manifold (see [20])
- the basic cohomology must be finite-dimensional (see [14], [28], [21])
- for any metric on the manifold, the orthogonal projection
$$P : L^2\left(\Omega\left(M\right)\right) \to L^2\left(\Omega\left(M, \mathcal{F}\right)\right)$$
must map the subspace of smooth forms onto the subspace of smooth basic forms ([21]).

Riemannian foliations were introduced by B. Reinhart in 1959 ([24]). Good references for Riemannian foliations and bundle-like metrics include the books and papers of B. Reinhart, F. W. Kamber, Ph. Tondeur, P. Molino, for example.

1.3. **The basic Laplacian.** Many researchers have studied basic forms and the basic Laplacian on Riemannian foliations with bundle-like metrics (see [1], [14], [28]). The basic Laplacian Δ_b for a given bundle-like metric is a version of the Laplace operator that preserves the basic forms and that is essentially self-adjoint on the L^2-closure of the space of basic forms. We define the basic Laplacian Δ_b by

$$\Delta_b = d\delta_b + \delta_b d : \Omega\left(M, \mathcal{F}\right) \to \Omega\left(M, \mathcal{F}\right),$$

where δ_b is the L^2-adjoint of the restriction of d to basic forms: $\delta_b = P\delta$ is the ordinary adjoint of d followed by the orthogonal projection onto the space of basic forms.

The operator Δ_b and its spectrum depend on the choice of the bundle-like metric and provide invariants of that metric. See [12], [16], [17], [21], [25], [26] for results. One may think of this operator as the Laplacian on the space of leaves. This operator is the appropriate one for physical intuition. For example, the Laplacian is used in the heat equation, which determines the evolution of the temperature distribution over a manifold as a function of time. If we assume that the leaves of the foliation are perfect conductors of heat, then the basic Laplacian is the appropriate operator that allows one to solve the heat distribution problem in this situation.

It turns out that the basic Laplacian is the restriction to basic forms of a second order elliptic operator on all forms, and this operator is not necessarily symmetric ([21]). Only in special cases is this operator the same as the ordinary Laplacian.

The basic Laplacian Δ_b is also not the same as the formal Laplacian defined on the local quotient manifolds of the foliation charts (or on a transversal). This transversal Laplacian is in general not symmetric on the space of basic forms, but it does preserve $\Omega(M, \mathcal{F})$.

The basic heat flow asymptotics are more complicated than that of the standard heat kernel, but there is a fair amount known (see [21], [25], [26]).

1.4. **The basic adjoint of the exterior derivative and mean curvature.** We assume $(M, \mathcal{F}, g_M)$ is a Riemannian foliation with bundle-like metric compatible with the Riemannian structure $(M, \mathcal{F}, g_Q)$. For later use, we define the mean curvature one-form κ and discuss the operator $\kappa_b \lrcorner$. Let

$$H = \sum_{i=1}^{p} \pi\left(\nabla^M_{f_i} f_i\right),$$

where $\pi : TM \to N\mathcal{F}$ is the bundle projection and $(f_i)_{1\leq i\leq p}$ is a local orthonormal frame of $T\mathcal{F}$. This is the mean curvature vector field, and its dual one-form is $\kappa = H^\flat$. Let $\kappa_b = P\kappa$ be the (smooth) basic projection of this mean curvature one-form. Let $\kappa_b\lrcorner$ denote the (pointwise) adjoint of the operator $\kappa_b\wedge$. Clearly, $\kappa_b\lrcorner$ depends on the choice of bundle-like metric g_M, not simply on the transverse metric g_Q.

It turns out that κ_b is a closed form whose cohomology class in $H^1(M, \mathcal{F})$ is independent of the choice of bundle-like metric (see [1]).

Recall the following expression for δ_b, the L^2-adjoint of d restricted to the space of basic forms of a particular degree (see [28], [21]):

$$\begin{aligned} \delta_b &= P\delta \\ &= \pm\bar{*}d\bar{*} + \kappa_b\lrcorner \\ &= \delta_T + \kappa_b\lrcorner, \end{aligned}$$

where

- δ_T is the formal adjoint (with respect to g_Q) of the exterior derivative on the transverse local quotients.
- the pointwise transversal Hodge star operator $\bar{*}$ is defined on all k-forms γ by

  $$\bar{*}\gamma = (-1)^{p(q-k)} * (\gamma \wedge \chi_{\mathcal{F}}),$$

  with $\chi_{\mathcal{F}}$ being the leafwise volume form, the characteristic form of the foliation and $*$ being the ordinary Hodge star operator. Note that $\bar{*}^2 = (-1)^{k(q-k)}$ on k-forms.
- The sign $\pm$ above only depends on dimensions and the degree of the basic form.

1.5. **Twisted duality for basic cohomology.** Even for transversally oriented Riemannian foliations, Poincaré duality does not necessarily hold for basic cohomology.

However, note that $d-\kappa_b\wedge$ is also a differential which defines a cohomology of basic forms. That is, since $d(\kappa_b)=0$, it follows from the Leibniz rule that $(d-\kappa_b\wedge)^2=0$ as an operator on forms, and it maps basic forms to basic forms. This differential also has the property that

$$\delta_b \bar{*}\alpha = (-1)^{k+1}\,\bar{*}\,(d-\kappa_b\wedge)\,\alpha$$

on every basic k-form α. As a result, the transversal Hodge star operator implements an isomorphism between different kinds of basic cohomology groups (see [13], [28], and [21]):

$$H_d^*(M,\mathcal{F}) \cong H_{d-\kappa_b\wedge}^{q-*}(M,\mathcal{F}).$$

This is called *twisted Poincaré duality.*

1.6. **Ordinary Dirac operators and examples.** See a reference such as [27] for the well-known details for this section. The ordinary Dirac operator in Euclidean space is given by

$$D=\sum e_k\cdot\frac{\partial}{\partial x_k},$$

where the operators $e_k\cdot$ are multiplication by matrices satisfying the relation

$$e_k\cdot e_j\cdot\ + e_j\cdot e_k\cdot\ = -2\delta_{kj}\mathbf{1}.$$

In three dimensions, the matrices can be chosen to be the Pauli spin matrices

$$e_1\cdot\ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ e_2\cdot\ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ e_3\cdot\ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

These relations are the same as the relations of the complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ associated to a vector space V. In general what is needed to define an ordinary Dirac operator on a Riemannian manifold M is a vector bundle $E\to M$ that is a bundle of $\mathbb{C}\mathrm{l}(TM)$ Clifford modules with compatible connection ∇^E. The *Dirac operator* D is the composition of the maps

$$\Gamma(E)\xrightarrow{\nabla^E}\Gamma(T^*M\otimes E)\xrightarrow{\cong}\Gamma(TM\otimes E)\xrightarrow{\mathrm{Cliff}}\Gamma(E),$$

where the last map is Clifford multiplication, denoted by "$\cdot$". We may write

$$D=\sum e_i\cdot\nabla_{e_i}^E$$

acting on $\Gamma(E)$, where (e_i) is a local orthonormal frame of TM. Computations show that D is elliptic, essentially self-adjoint, and thus has discrete spectrum.

Examples of the Dirac operator include:

- "The" spin^c Dirac operator. Here E is a spinor bundle, which at each point $p\in M$ is an irreducible representation space for $\mathbb{C}\mathrm{l}(T_pM)$.
- The de Rham operator

$$d+\delta:\Omega^{\mathrm{even}}(M)\to\Omega^{\mathrm{odd}}(M).$$

- The signature operator

$$d+\delta:\Omega^{+}(M)\to\Omega^{-}(M),$$

where the $\pm$ refer to self-dual and anti-self-dual forms. There is an operator of the form $\bigstar = i^{k(k-1)+n}*$ acting on complex-valued k-forms on a $2n$-dimensional manifold, and $\bigstar^2 = \mathbf{1}$. The space Ω^+ of self-dual forms is the $+1$ eigenspace of $\bigstar$, and the space Ω^- of antiself-dual forms is the -1-eigenspace of $\bigstar$. It turns out that $d+\delta$ anticommutes with $\bigstar$ and thus maps $\Omega^\pm$ to $\Omega^\mp$.

- The Dolbeault operator

$$\partial + \overline{\partial} : \Omega^{0,\text{even}}(M) \to \Omega^{0,\text{odd}}(M).$$

Each one of these operators has an associated Laplacian D^2 and associated harmonic forms, and the index of each of these operators ($\operatorname{index}(D) = \dim\ker D - \dim\ker D^*$) is an important topological invariant. For example, if $D = d+\delta$ is the de Rham operator, we have

$$\ker(d+\delta) = \mathcal{H},$$

the space of harmonic forms, which by Hodge theory can be used to represent the different cohomology classes. Thus,

$$\begin{aligned} \operatorname{index}(d+\delta)|_{\Omega^{\text{even}}} &= \dim\ker(d+\delta)|_{\Omega^{\text{even}}} - \dim\ker(d+\delta)|_{\Omega^{\text{odd}}} \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of M.

1.7. **The basic Dirac operator and statement of the main theorem.** We now discuss the construction of the basic Dirac operator (see [6], [8], [22], [3]), a construction which requires a choice of bundle-like metric. Let $(M,\mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation. Let $E \to M$ be a foliated vector bundle (see [15]) that is a bundle of $\mathbb{C}\mathrm{l}(Q)$ Clifford modules with compatible connection ∇^E. The *transversal Dirac operator* D_{tr} is the composition of the maps

$$\Gamma(E) \xrightarrow{(\nabla^E)^{\text{tr}}} \Gamma(Q^*\otimes E) \xrightarrow{\cong} \Gamma(Q\otimes E) \xrightarrow{\text{Cliff}} \Gamma(E),$$

where the last map denotes Clifford multiplication, denoted by "$\cdot$", and the operator $(\nabla^E)^{\text{tr}}$ is the projection of ∇^E. The transversal Dirac operator fixes the basic sections $\Gamma_b(E) \subset \Gamma(E)$ (i.e. $\Gamma_b(E) = \{s \in \Gamma(E) : \nabla^E_X s = 0 \text{ for all } X \in \Gamma(L)\}$) but is not symmetric on this subspace. By modifying D_{tr} by a bundle map, we obtain a symmetric and essentially self-adjoint operator D_b on $\Gamma_b(E)$. Let κ_b be the L^2-orthogonal projection of κ onto the space of basic forms as explained above, and let $\kappa_b^\sharp$ be the corresponding vector field. We now define

$$\begin{aligned} D_{\text{tr}}\, s &= \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s\ , \\ D_b s &= \frac{1}{2}(D_{\text{tr}} + D^*_{\text{tr}})s = \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s - \frac{1}{2}\kappa_b^\sharp \cdot s\ , \end{aligned}$$

where $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q. A direct computation shows that D_b preserves the basic sections, is transversally elliptic, and thus has discrete spectrum ([8], [6], [10]).

An example of the basic Dirac operator is as follows. Using the bundle $\wedge^* Q$ as the Clifford bundle with Clifford action $e\cdot\ = e^*\wedge - e^*\lrcorner$ in analogy to the ordinary de Rham operator, we have

$$\begin{aligned} D_{\mathrm{tr}} &= d+\delta_T = d+\delta_b-\kappa_b\lrcorner : \Omega^{\mathrm{even}}(M,\mathcal{F}) \to \Omega^{\mathrm{odd}}(M,\mathcal{F}) \\ D_b &= \frac{1}{2}(D_{\mathrm{tr}}+D_{\mathrm{tr}}^*)s = d+\delta_b-\kappa_b\lrcorner - \frac{1}{2}\kappa_b^\sharp\cdot \\ &= d+\delta_b-\kappa_b\lrcorner-\frac{1}{2}\left(\kappa_b\wedge -\kappa_b\lrcorner\right) \\ &= d+\delta_b-\frac{1}{2}\kappa_b\lrcorner-\frac{1}{2}\kappa_b\wedge . \end{aligned}$$

One might have instead guessed that $d+\delta_b$ is the basic de Rham operator in analogy to the ordinary de Rham operator, for this operator is essentially self-adjoint, and the associated basic Laplacian yields basic Hodge theory that can be used to compute the basic cohomology.

We study the invariance of the spectrum of the basic Dirac operator with respect to a change of bundle-like metric; that means when one modifies the metric on M in any way that leaves the transverse metric on the normal bundle intact (this includes modifying the subbundle $N\mathcal{F}\subset TM$, as one must do in order to make the mean curvature basic, for example). In [10], we prove

Theorem 1.1. *Let $(M,\mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E\to M$. The spectrum of the basic Dirac operator is the same for every possible choice of bundle-like metric that is associated to the transverse metric on the quotient bundle Q.*

We emphasize that the basic Dirac operator D_b depends on the choice of bundle-like metric, not merely on the Clifford structure and Riemannian foliation structure, since both projections $T^*M\to Q^*$ and P_b as well as κ_b depend on the leafwise metric.

2. Proof of the main theorem

The proof of the main theorem is contained in [10]. The idea of proof is as follows. One can show that every different choice of bundle-like metric changes the L^2-inner product by multiplication by a specific smooth, positive basic function. This changes the basic Dirac operator by a zeroth order operator that is Clifford multiplication by an exact basic one-form. This new operator is conjugate to the original one, and thus the spectrum of the operator is independent of the metric choice.

3. Consequences of the main theorem

D. Dominguez showed that every Riemannian foliation admits a bundle-like metric for which the mean curvature form is basic [5]. Further, the bundle-like metric may be chosen so that the mean curvature is basic-harmonic (in the new metric); see [18] and [19]. Therefore, in calculating or estimating the eigenvalues of the basic Dirac operator, one may choose the bundle-like metric so that the mean curvature is basic-harmonic. Immediately we may obtain stronger inequalities for eigenvalue estimates. We give one example below.

In [11], S. D. Jung showed that the eigenvalues λ of the basic Dirac operator on spin foliations (where the normal bundle carries a spin structure as a foliated vector bundle) satisfy

$$\lambda^2 \geq \frac{q}{4(q-1)}\inf{}_M(\mathrm{Scal}^\nabla+|\kappa|^2),$$

under the assumption that the mean curvature form κ is basic and basic-harmonic. In [9], the author obtained another Friedrich-type estimate [4] for the eigenvalues of the basic Dirac operator for bundle-like metrics with basic-harmonic mean curvature. Since by Theorem 1.1 the spectrum does not change, we may improve both results to deduce that for **any** bundle-like metric the eigenvalues of the basic Dirac operator satisfy

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(\mathrm{Scal}^\nabla). \tag{3.1}$$

This estimate is of interest only for positive transversal scalar curvature. Moreover, in [10] we show

Proposition 3.1. *Let $(M, \mathcal{F})$ be a compact Riemannian manifold endowed with a spin foliation with basic mean curvature κ. Then, we have the estimate*

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(\mathrm{Scal}_M - \mathrm{Scal}_L + |A|_Q^2 + |T|_L^2).$$

If $\mathcal{F}$ is a Riemannian flow (i.e. $p = 1$), then

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(\mathrm{Scal}_M + |A|_Q^2 + |\kappa|^2). \tag{3.2}$$

If the limiting case is attained, the foliation is minimal and we have a transversal Killing spinor.

Here A and T denote the O'Neill tensors [2, 23] of the foliation.

More applications can be found in the paper [10].

4. Modified differentials, Laplacians, and basic cohomology

From the above, the basic de Rham operator is

$$\begin{aligned} D_b &= d + \delta_b - \frac{1}{2}\kappa_b \lrcorner - \frac{1}{2}\kappa_b \wedge \\ &= \widetilde{d} + \widetilde{\delta} \end{aligned}$$

acting on basic forms, where

$$\widetilde{d} = d - \frac{1}{2}\kappa_b \wedge, \widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b \lrcorner.$$

The operators $\widetilde{d}$ and $\widetilde{\delta}$ have interesting properties:

- They are differentials: $\widetilde{d}^2 = 0$, $\widetilde{\delta}^2 = 0$.
- $\widetilde{\delta}\overline{*} = \pm\overline{*}\widetilde{d}$.

From the first property we see that we can define cohomology using these differentials. From the second property, we know that we may define a basic signature operator. This was not known and not possible previously with ordinary basic cohomology and the operator $d + \delta_b$, because that operator would not map self-$\overline{*}$-dual basic forms to anti-self-$\overline{*}$-dual forms. The eigenvalues of this operator depend only on the Riemannian foliation structure.

The new basic cohomology defined using $\widetilde{d}$ will satisfy Poincaré duality, and the isomorphism is implemented using $\overline{*}$. Also, even though the differential depends on the choice of the bundle-like metric, the dimensions of the resulting cohomology groups are independent of that choice. The consequences of these results will be explained in a forthcoming paper.

References

[1] J. A. Álvarez-López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. **10** (1992), 179–194.

[2] A. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb 3, Folge 10, Springer-Verlag, Berlin 1987.

[3] J. Brüning, F. W. Kamber, and K. Richardson, *The eta invariant and equivariant index of transversally elliptic operators*, preprint in preparation.

[4] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannshen Mannigfaltigkeit nichnegativer Skalar-krümmung*, Math. Nachr. **97** (1980), 117–146.

[5] D. Domínguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci. Sér. I **320** (1995), 1331-1335.

[6] R. G. Douglas, J. F. Glazebrook, F. W. Kamber, and G. L. Yu, *Index formulas for geometric Dirac operators in Riemannian foliations*, K-Theory **9** (1995), no. 5, 407–441.

[7] A. El Kacimi-Alaoui, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), 57–106.

[8] J. F. Glazebrook and F. W. Kamber, *Transversal Dirac families in Riemannian foliations*, Comm. Math. Phys. **140** (1991), 217-240.

[9] G. Habib, *Tenseur d'impulsion-énergie et feuilletages*, Ph.D thesis (2006), Université Henri Poincaré, Nancy.

[10] G. Habib and K. Richardson, *A brief note on the spectrum of the basic Dirac operator*, Bull. London Math. Soc. **41**(2009), 683-690.

[11] S.D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. **39** (2001), 253–264.

[12] S. D. Jung and K. Richardson, *Transverse conformal Killing forms and a Gallot-Meyer Theorem for foliations*, preprint arXiv:0805.4187v1 [math.DG].

[13] F. W. Kamber and Ph. Tondeur, *Duality theorems for foliations*, Transversal structure of foliations (Toulouse, 1982), Astérisque **116** (1984), 108–116.

[14] F. W. Kamber and Ph. Tondeur, *De Rham-Hodge theory for Riemannian foliations*, Math. Ann. **277** (1987), 415–431.

[15] F. W. Kamber and Ph. Tondeur, *Foliated bundles and characteristic classes*, Lecture Notes in Math. **493**, Springer-Verlag, Berlin–New York, 1975.

[16] J. Lee and K. Richardson, *Lichnerowicz and Obata theorems for forliations*, Pacific J. Math. 206 (2002), 339-357.

[17] J. Lee and K. Richardson, *Riemannian foliations and eigenvalue comparison*, Ann. Global Anal. Geom. **16** (1998), 497-525.

[18] P. March, M. Min-Oo, E. Ruh, *Mean curvature of Riemannian foliations*, Canad. Math. Bull. **39** (1996), no. 1, 95–105.

[19] A. Mason, *An application of stochastic flows to Riemannian foliations*, Houston J. Math. 26 (2000), 481-515.

[20] P. Molino, *Riemannian foliations*, Progress in Mathematics **73**, Birkhäuser, Boston 1988.

[21] E. Park and K. Richardson, *The basic Laplacian of a Riemannian foliation*, Amer. J. Math. **118** (1996), 1249–1275.

[22] I. Prokhorenkov and K. Richardson, *Natural Equivariant Dirac Operators*, preprint arXiv:0805.3340v1 [math.DG].

[23] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

[24] B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. Math. **69** (1959), 119–132.

[25] K. Richardson, *Asymptotics of heat kernels on Riemannian foliations*, Geom. Funct. Anal. **8** (1998), 356–401.

[26] K. Richardson, *Traces of heat operators on Riemannian foliations*, to appear in Trans. A.M.S., available at arXiv:0710.1324v1 [math.DG].

[27] J. Roe, *Elliptic operators, topology and asymptotic methods*, 2nd ed., Pitman Research Notes in Mathematics Series **395**, Harlow, England: Longman, 1998.

[28] Ph. Tondeur, *Geometry of foliations*, Monographs in Mathematics **90**, Birkhäuser Verlag, Basel 1997.

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA
E-mail address, K. Richardson: k.richardson@tcu.edu