

DESINGULARIZING COMPACT LIE GROUP ACTIONS

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ABSTRACT. This note surveys the well-known structure of G -manifolds and summarizes parts of two papers that have not yet appeared: [4], joint with J. Brüning and F. W. Kamber, and [8], joint with I. Prokhorenkov. In particular, from a given manifold on which a compact Lie group acts smoothly, we construct a sequence of manifolds on which the same Lie group acts, but with fewer levels of singular strata. Global analysis and geometric results on the simpler manifolds may be translated to results on the original manifold. Further, we show that by utilizing bundles over a G -manifold with singular strata, we may construct natural equivariant transverse Dirac-type operators that have properties similar to Dirac operators on closed manifolds.

1. MANIFOLDS AND COMPACT LIE GROUP ACTIONS

Suppose that a compact Lie group G acts smoothly on a smooth, connected, closed manifold M . We assume that the action is effective, meaning that no $g \in G$ fixes all of M . (Otherwise, replace G with $G / \{g \in G : gx = x \text{ for all } x \in M\} = \emptyset$.) Choose a Riemannian metric for which G acts by isometries; average the pullbacks of any fixed Riemannian metric over the group of diffeomorphisms to obtain such a metric.

Given such an action and $x \in M$, the **isotropy subgroup** $G_x < G$ is defined to be $\{g \in G : gx = x\}$. The **orbit** \mathcal{O}_x of a point x is defined to be $\{gx : g \in G\}$. Note that $G_{gx} = gG_xg^{-1}$, so the conjugacy class of the isotropy subgroup of a point is fixed along an orbit. The conjugacy class of the isotropy subgroups along an orbit is called the **orbit type**. On any such G -manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G that occur as isotropy subgroups, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K , and we say $[H] < [K]$ if $[H] \leq [K]$ and $[H] \neq [K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0], \dots, [G_{r-1}]$ such that $[G_i] \leq [G_j]$ if and only if $i \leq j$. It is well-known that the union M_0 of the **principal orbits** (those with type $[G_0]$) form an open dense subset M_0 of the manifold M , and the other orbits are called **singular**. As a consequence, every isotropy subgroup H satisfies $[G_0] \leq [H]$; M_0 is called the **principal stratum**. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j ; the set M_j is called the **stratum** corresponding to $[G_j]$. A stratum M_j is called a **most singular stratum** if there does not exist a stratum M_k such that $[G_j] < [G_k]$. It is known that each stratum is a G -invariant submanifold of M , and in fact a most singular stratum is a closed (but not necessarily connected) submanifold. Also, for each j , the submanifold $M_{\geq j} := \bigcup_{[G_k] \geq [G_j]} M_k$ is a closed, G -invariant submanifold.

Consider the following simple examples.

Example 1.1. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $M = S^2 \subset \mathbb{R}^3$ by

$$(1, 0)(x, y, z) = (x, -y, z), \quad (0, 1)(x, y, z) = (x, y, -z).$$

The action is isometric for the standard metric on the sphere. The isotropy subgroup of each point in $M_0 = \{(x, y, z) \in S^2 : z \neq 0, y \neq 0\}$ is $G_0 = \{(0, 0)\}$, points of the set $M_2 = \{(x, y, z) \in S^2 : z = 0, y \neq 0\}$ have isotropy subgroup $G_2 = \{(0, 0), (0, 1)\}$, points in $M_1 = \{(x, y, z) \in S^2 : y = 0, z \neq 0\}$ have isotropy subgroup $G_1 = \{(0, 0), (1, 0)\}$, and points of $M_3 = \{(1, 0, 0), (-1, 0, 0)\}$ have isotropy subgroup $G_3 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here, M_0 is the principal stratum, and M_3 is the only most singular stratum. The set $M_{\geq 1}$ is the circle $\{(x, y, z) \in S^2 : y = 0\}$. Note that $[G_0] < [G_1] < [G_3]$ and $[G_0] < [G_2] < [G_3]$ but $[G_1]$ and $[G_2]$ are not comparable.

Example 1.2. $G = S^1$ acts on $M = S^2 \subset \mathbb{R}^3$ by

$$e^{i\theta}(x, y, z) = (x \cos(\theta) - y \sin(\theta), x \sin(\theta) + y \cos(\theta), z),$$

i.e. rotations around the z -axis. This action is isometric for the standard metric. The isotropy subgroup at each point of $M_0 = \{(x, y, z) \in S^2 : |z| < 1\}$ is $G_0 = \{1\}$, and the isotropy subgroup for each point of $M_1 = \{(x, y, z) \in S^2 : z = 1 \text{ or } z = -1\}$ is $G_1 = S^1$.

Example 1.3. $G = S^1 \times S^1$ acts on $M = S^3 \subset \mathbb{R}^4$ by

$$(e^{i\theta}, e^{i\alpha})(x, y, z, w) = (x \cos(\theta) - y \sin(\theta), x \sin(\theta) + y \cos(\theta), z \cos(\alpha) - w \sin(\alpha), z \sin(\alpha) + w \cos(\alpha)).$$

The isotropy subgroup of each point of

$$M_0 = \{(x, y, z, w) \in S^3 : x^2 + y^2 > 0, z^2 + w^2 > 0\}$$

is $G_0 = \{(1, 1)\}$, the isotropy subgroup for each point of $M_1 = \{(x, y, z, w) \in S^3 : z = w = 0\}$ is $G_1 = \{1\} \times S^1$, and the isotropy subgroup of $M_2 = \{(x, y, z, w) \in S^3 : x = y = 0\}$ is $G_2 = S^1 \times \{1\}$. Note that the diagonal action of $(e^{i\theta}, e^{i\theta})$ with $e^{i\theta} \in S^1$ gives the Hopf fibration. In this example, the torus action on S^3 yields three strata, with M_0 being the principal stratum and M_1 and M_2 each being a most singular stratum.

2. DESINGULARIZATION CONSTRUCTION

With notation as in the previous section, we will construct a new G -manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M , branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the desingularization of M and is denoted \widetilde{M} . The significance of this construction is that it appears in the equivariant index theorem in [4], and the analysis of transversally elliptic operators on M may be replaced by analysis on \widetilde{M} , which is much easier to understand.

A sequence of constructions is used to construct N and $\widetilde{M} \subset N$. If M_j is a most singular stratum, let $T_\varepsilon(M_j)$ denote an open tubular neighborhood of M_j of radius $\varepsilon > 0$. If ε is sufficiently small, then all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary. Since the $T_\varepsilon(M_j)$ is saturated (a union of G -orbits), the G -action lifts to N^1 . Note that the strata of the G -action on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1 = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G -manifold with one fewer stratum than M , and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted \widetilde{M}^1 in N^1 . In fact, N^1 is a branched double cover of M , branched over M_j . If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G -manifold N^1 to produce a new G -manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M . Again, \widetilde{M}^2 is a fundamental domain of $\widetilde{M}^1 \setminus \{\text{a most singular stratum}\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^{r-1}$ is a G -manifold with all orbits of type $[G_0]$ and is a 2^{r-1} -fold branched cover of M , branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^{r-1}$, which is a fundamental domain of M_0 in N .

As mentioned earlier, if M is equipped with a G -equivariant, transversally elliptic differential operator on sections of an equivariant vector bundle over M , then this data may be pulled back to the desingularization \widetilde{M} . Given the bundle and operator over N^j , simply form the invertible double of the operator on N^{j+1} , which is the double of the manifold with boundary $N^j \setminus T_\varepsilon(\Sigma)$, where Σ is a most singular stratum on N^j .

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G -manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. Note that the isotropy type of all points of $\widetilde{M}_{\geq j}$ is $[G_j]$, and the $\widetilde{M}_{\geq j}/G$ is a smooth (open) manifold.

The desingularizations \widetilde{M} and $\widetilde{M}_{\geq j}$ are the regions of integration present in the following equivariant index formula in [4].

Theorem 2.1. (*Equivariant Index Theorem, in [4]*)

$$\begin{aligned} \text{ind}^\rho(D) &= \int_{\widetilde{M}/G} A_0^\rho(x) |\widetilde{dx}| \\ &+ \sum_{j,a,b} C_{jab} \int_{\widetilde{M}_{\geq j}/G} \left(-\eta\left(D_j^{S^+, \sigma_a}\right) + h\left(D_j^{S^+, \sigma_a}\right) \right) A_{j, \sigma_b^*}^{\rho_0}(x) |\widetilde{dx}|. \end{aligned}$$

In this formula, D is a G -equivariant, transversally elliptic operator which can be written in a tubular neighborhood of each M_j as the product

$$D = \left\{ Z_j \left(\nabla_{\partial_r}^E + \frac{1}{r} D_j^S \right) \right\} * D^{M_{\geq j}},$$

where r is the distance from $M_{\geq j}$, where Z_j is a local bundle isomorphism, the map D_j^S is a family of purely first order operators that differentiates in the unit normal bundle directions tangent to $S_x M_{\geq j}$, and $D^{M_{\geq j}}$ is a global transversally elliptic, G -equivariant, first order operator on the stratum $M_{\geq j}$. Many important examples of equivariant, transversally elliptic differential operators satisfy the condition above. In the formula, the forms A_0^ρ and $A_{j, \sigma_b^*}^{\rho_0}$ are Atiyah-Singer integrands for differential operators on the corresponding desingularized strata. The number $\eta\left(D_j^{S^+, \sigma_a}\right)$ corresponds to a G_j -equivariant eta invariant, and $h\left(D_j^{S^+, \sigma_a}\right)$ is the dimension of the σ_a -part of the kernel of $D_j^{S^+}$, which is a G_j representation space. These numbers are actually topological invariants, a little surprising if one is familiar with eta invariants of operators on closed manifolds; the invariance comes from the fact that the polar decomposition yields integral eigenvalues for D_j^S , and thus they are constant along connected components of M_j . The constants C_{jab} depend on the representation theory — the

induced representation of G_j on a ρ -representation space and on Clebsch-Gordan coefficients of tensor product representations.

3. NATURAL EQUIVARIANT DIRAC OPERATORS

Here another approach is used to treat difficult transverse analytic problems in a less singular setting, and the content of this section is in the paper [8]. Given a connected, complete G -manifold, the action of $g \in G$ on M induces an action of dg on TM , which in turn induces an action of G on the principal $O(n)$ -bundle $F_O \xrightarrow{p} M$ of orthonormal frames over M . It turns out that when G is effective, the isotropy subgroups on F_O are all trivial. In any case, the G orbits on F_O are diffeomorphic and are the fibers (leaves) of a Riemannian fiber bundle in a natural Sasakian metric on F_O . The quotient $F_O \xrightarrow{\pi} F_O/G$ is a Riemannian submersion of compact $O(n)$ -manifolds. The metric on F_O is bundle-like for the Riemannian foliation (F_O, \mathcal{F}) by G -orbits.

Let $E \rightarrow F_O$ be a Hermitian vector bundle that is equivariant with respect to the $G \times O(n)$ action. Let $\rho : G \rightarrow U(V_\rho)$ and $\sigma : O(n) \rightarrow U(W_\sigma)$ be irreducible unitary representations. We define the bundle $\mathcal{E}^\sigma \rightarrow M$ by

$$\mathcal{E}_x^\sigma = \Gamma(p^{-1}(x), E)^\sigma,$$

where the superscript σ refers to the type σ part of the $O(n)$ -representation space $\Gamma(p^{-1}(x), E)$. The bundle \mathcal{E}^σ is a Hermitian G -vector bundle of finite rank over M that comes with a natural metric.

Similarly, we define the bundle $\mathcal{T}^\rho \rightarrow F_O/G$ by

$$\mathcal{T}_y^\rho = \Gamma(\pi^{-1}(y), E)^\rho,$$

and $\mathcal{T}^\rho \rightarrow F_O/G$ is a Hermitian $O(n)$ -equivariant bundle of finite rank, with a natural metric.

Theorem 3.1. *(in [8]) For any irreducible representation $\rho : G \rightarrow U(V_\rho)$ and irreducible representation $\sigma : O(n) \rightarrow U(W_\sigma)$, there is an explicit isomorphism $\Gamma(M, \mathcal{E}^\sigma)^\rho \rightarrow \Gamma(F_O/G, \mathcal{T}^\rho)^\sigma$ that extends to an L^2 -isometry.*

Next, let $E \rightarrow F_O$ be a Hermitian vector bundle of $\mathbb{C}l(N\mathcal{F})$ modules that is equivariant with respect to the $G \times O(n)$ action. As explained in [5], [6], [8], [4], [7], there is natural construction of what is known as the **basic Dirac operator** in this situation. We have the transversal Dirac operator D_{tr} defined by the composition

$$\Gamma(F_O, E) \xrightarrow{\nabla} \Gamma(F_O, T^*F_O \otimes E) \xrightarrow{\text{proj}} \Gamma(F_O, N^*\mathcal{F} \otimes E) \xrightarrow{c} \Gamma(F_O, E).$$

The basic Dirac operator

$$D_{N\mathcal{F}} = \frac{1}{2}(D_{\text{tr}} + D_{\text{tr}}^*) = D_{\text{tr}} - \frac{1}{2}c(H)$$

is an essentially self-adjoint $G \times O(n)$ -equivariant operator, where H is the mean curvature vector field of the G -orbits in F_O .

From $D_{N\mathcal{F}}$ we now construct equivariant differential operators on M and F_O/G , denoted

$$D_M^\sigma : \Gamma(M, \mathcal{E}^\sigma) \rightarrow \Gamma(M, \mathcal{E}^\sigma)$$

and

$$D_{F_O/G}^\rho : \Gamma(F_O/G, \mathcal{T}^\rho) \rightarrow \Gamma(F_O/G, \mathcal{T}^\rho),$$

defined in the natural way. For an irreducible representation $\alpha : G \rightarrow U(V_\alpha)$, let

$$(D_M^\sigma)^\alpha : \Gamma(M, \mathcal{E}^\sigma)^\alpha \rightarrow \Gamma(M, \mathcal{E}^\sigma)^\alpha$$

be the restriction of D_M^σ to sections of G -representation type $[\alpha]$. Similarly, for an irreducible representation $\beta : G \rightarrow U(W_\beta)$, let

$$(D_{F_O/G}^\rho)^\beta : \Gamma(F_O/G, \mathcal{T}^\rho)^\beta \rightarrow \Gamma(F_O/G, \mathcal{T}^\rho)^\beta$$

be the restriction of $D_{F_O/G}^\rho$ to sections of $O(n)$ -representation type $[\beta]$. The proposition below follows from Theorem 3.1.

Proposition 3.2. *The operator D_M^σ is transversally elliptic and G -equivariant, and $D_{F_O/G}^\rho$ is elliptic and $O(n)$ -equivariant, and the closures of these operators are self-adjoint. The operators $(D_M^\sigma)^\rho$ and $(D_{F_O/G}^\rho)^\sigma$ have identical discrete spectrum, and the corresponding eigenspaces are conjugate via Hilbert space isomorphisms.*

Thus, questions about the transversally elliptic operator D_M^σ are reduced to questions about the elliptic operators $D_{F_O/G}^\rho$ for each irreducible $\rho : G \rightarrow U(V_\rho)$. Further, it turns out that the operators D_M^σ play the same role for equivariant analysis as the standard Dirac operators do in the index theory and analysis of elliptic operators on closed manifolds.

4. FURTHER COMMENTS

The main use of these results is that in both cases, a quite difficult problem of analyzing a transversally elliptic operator (with potentially infinite dimensional eigenspaces) is reduced to an elliptic problem or set of elliptic problems, which are much more tractable. For example, the Atiyah-Segal Theorem ([1]) was the first version of an equivariant index theorem, and it appeared in 1968. However, the appropriate generalization to transversally elliptic operators appeared only in 1996 and was due to Berline and Vergne ([2],[3]). In a sense, Theorem 2.1 is a Fourier transform version of the Atiyah-Segal and Berline-Vergne results, giving a formula for the Fourier coefficients of the character instead of the value of the character at a particular $g \in G$. Further, Theorem 2.1 gives a method of computing eta invariants of Dirac-type operators on quotients of spheres by compact group actions; this was known previously for finite group actions only.

The new “transversal Dirac operators” on G -manifolds constructed in Section 3 and in [8] should be explored further, and in particular future investigations should lead to generalizations of Dirac operator results to the transversally elliptic setting.

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