Spectral Geometry of G-manifolds and foliations

---

by

Ken Richardson

Soon available at
http://www.math.tcu.edu/math/Preprints.html

G-manifolds

Let G be a compact Lie group that acts on the right by isometries on a compact, n-dimensional Riemannian manifold M. The pair (M, G) along with the action is called a G-manifold. The orbits xG of this action are compact, homogeneous submanifolds. These submanifolds are locally equidistant but do not necessarily have constant dimension. The Lie derivative of the metric is zero in directions tangent to the orbits.

Riemannian foliations

Suppose a compact, Riemannian manifold M' is endowed with a transversally oriented, codimension-k foliation $\mathcal{F}$ — a partition of M' into immersed, transversally oriented submanifolds (leaves) of codimension k, such that the leaves are locally equidistant. Such a metric is called *bundle-like for* $\mathcal{F}$, and $(M', \mathcal{F})$ is called a *Riemannian foliation*. The Lie derivative of the transverse metric is zero in directions tangent to the leaves. The leaf closures are compact submanifolds.

bundle-like	NOT bundle-like

Theorem 1. *Let G be a compact Lie group that acts on the right by isometries on a compact, n-dimensional Riemannian manifold M. Then exists a Riemannian foliation $(M', \mathcal{F})$of codimension n and an isometry $\Phi : M/G \to M'/\overline{\mathcal{F}}$ such that the volume of the leaf closure $\Phi(C)$ is the same as the volume of the orbit $C \in M/G$.*

Proof. Choose a finite subset $S \subset G$ so that the subgroup Γ generated by S is dense in G. Let X be any compact Riemannian manifold with volume 1 such that there is a surjective homomorphism $\mu : \pi_1(X) \to \Gamma$. Let $M' = \widetilde{X} \times M/\sim$, where $(x, y) \sim (x[\gamma], y\,\mu([\gamma]))$ for every $x \in \widetilde{X}$, $y \in M$, and $[\gamma] \in \pi_1(X)$. Let the foliation $\mathcal{F}$ be defined by letting the leaves be sets of the form $L_{y_0} = \left\{[(x, y_0)]_\sim \mid x \in \widetilde{X}\right\}$. □

Theorem 2. *Let M' be a compact, Riemannian manifold, and let $(M', \mathcal{F})$ be a transversally oriented Riemannian foliation of codimension n. Then there is a compact $SO(n)$-manifold W and an isometry $\Theta : M'/\overline{\mathcal{F}} \to W/G$ such that* $\text{Vol}\left(\Theta\left(\overline{L}\right)\right) = \text{Vol}\left(\overline{L}\right)$ *for every leaf closure $\overline{L} \in M'/\overline{\mathcal{F}}$ that has finite holonomy (or, equivalently, maximal dimension).*

Proof. Let $\pi : \widehat{M} \to M'$ be the orthonormal transverse frame bundle of M', and use the normalized biinvariant metric on the fibers and the metric on M' to define a canonical metric on $\widehat{M}$. We lift the foliation $\mathcal{F}$ to a foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, and the closures of the lifted foliation form the fibers of a Riemannian submersion $\rho : \widehat{M} \to W$. Let $\Theta\left(\overline{L}\right) = \rho\left(\pi^{-1}\left(\overline{L}\right)\right)$. Next, modify the metric along the $SO\left(n\right)$-orbits in W so that the volume condition is satisfied. □

Consequences

Corollary 1. *Let G be any compact Lie group that acts by orientation-preserving isometries on a compact, oriented n-dimensional Riemannian manifold M. Then there exists a Riemannian $SO(n)$-manifold W such that $W/SO(n)$ is isometric to M/G via an isometry that preserves the volumes of orbits of maximal dimension.*

Corollary 2. *Let $(M, \mathcal{F})$ be a transversally oriented, codimension-q Riemannian foliation on a compact, Riemannian manifold. Then there exists a Riemannian manifold M' along with a transversally oriented Riemannian foliation $\mathcal{F}'$ on M' that is constructed by suspending an action of a subgroup of $SO(q)$, such that the leaf closure spaces $M/\overline{\mathcal{F}}$ and $M'/\overline{\mathcal{F}'}$ are isometric via an isometry that preserves volumes of leaf closures of maximal dimension.*

Laplacians

1. (Equivariant Laplacian of a G-manifold) Let M be a compact Riemannian G-manifold. The induced action of G on differential forms commutes with the Laplacian on M, and we define the *equivariant Laplacian* Δ_G to be the restriction of the ordinary Laplacian to equivariant forms — forms invariant under the G-action. Let the eigenvalues of Δ_G on G-invariant functions be denoted by

$$0 < \lambda_1^G \leq \lambda_2^G \leq \lambda_3^G \leq \ldots$$

Associated to Δ_G are the equivariant heat operator $e^{-t\Delta_G}$ and the equivariant heat kernel $K_G(t, x, y)$.

2. (Basic Laplacian of a Riemannian foliation) Let M' be a compact Riemannian manifold endowed with a Riemannian foliation $\mathcal{F}$. The basic functions of $(M', \mathcal{F})$ are the functions constant on the leaves of $\mathcal{F}$, and the basic forms are locally pullbacks of forms on $U/\mathcal{F}$, where U is a small open set in M'. Let the *basic Laplacian* Δ_B be defined by

$$\Delta_B = \delta_B d_B + d_B \delta_B,$$

where d_B is the restriction of d to basic forms, and δ_B is the adjoint of d_B on the space of basic forms using the L^2 inner product. Let the eigenvalues of Δ_B on functions be denoted by

$$0 < \lambda_1^B \leq \lambda_2^B \leq \lambda_3^B \leq \dots$$

Associated to Δ_B are the basic heat operator $e^{-t\Delta_B}$ and the basic heat kernel $K_B(t, x, y)$.

Equivariant Spectral Geometry

Let M be a compact G-manifold, and let $K(t,x,y)$ be the ordinary heat kernel on M. Then the equivariant heat kernel satisfies

$$K_G(t,x,y) = \int_G K(t,x,yg)\,dg,$$

where dg is the normalized Haar measure. Thus, the trace of the equivariant heat operator is

$$\begin{aligned} \mathrm{tr}\left(e^{-t\Delta_G}\right) &= \sum_{j=0}^{\infty} e^{-t\lambda_j^G} \\ &= \int_M \int_G K_G(t,x,xg)\,dg\,dV(x). \end{aligned}$$

- (Brüning, Heintze, 1984) The trace of the equivariant heat operator satisfies

$$\mathrm{tr}\left(e^{-t\Delta_G}\right) \sim (4\pi t)^{-m/2}\Bigg(a_0 + \sum_{\substack{j\geq 1\\ 0\leq k<K_0}} a_{jk}\, t^{j/2}(\log t)^k\Bigg) \text{ as } t\to 0,$$

  where $m = \dim M/G$, K_0 is less than or equal to the number of different dimensions of G-orbits in M, and $a_0 = \mathrm{Vol}(M/G)$. The coefficients a_{jk} depend only on the metrics on M and G and their derivatives on the subset $\{(g,x)\,|\,xg = x\} \subset G \times M$.

- (Corollary) The equivariant spectral counting function satisfies

$$N_G(\lambda) := \#\{\lambda_j^G | \lambda_j^G \leq \lambda\} \sim \frac{\mathrm{Vol}(M/G)}{(4\pi)^{m/2}\Gamma\left(\frac{m}{2}+1\right)}\lambda^{m/2}$$

  as $\lambda \to \infty$.

Theorem 3. *Let M' be a compact Riemannian manifold endowed with a transversally oriented, codimension-q Riemannian foliation $\mathcal{F}$. Then the basic heat kernel satisfies*

$$K_B(t,x',y') = K_{SO(q)}(t,x,y),$$

where $K_{SO(q)}(t,x,y)$ is the equivariant heat kernel associated to the the $SO(q)$ manifold W in Theorem 2, and x, respectively y, is any element of the $SO(q)$-orbit associated to the leaf closure containing x', respectively y'.

Theorem 4. *The trace of the basic heat operator satisfies*

$$\operatorname{tr}\left(e^{-t\Delta_B}\right) \sim (4\pi t)^{-\overline{q}/2}\Bigg(a_0 + \sum_{\substack{j\geq 1\\ 0\leq k<K_0}} a_{jk}\, t^{j/2}(\log t)^k\Bigg) \text{ as } t\to 0,$$

where $\overline{q} = \dim M'/\overline{\mathcal{F}}$, K_0 is less than or equal to the number of different leaf closure dimensions, and $a_0 = \operatorname{Vol}\left(M'/\overline{\mathcal{F}}\right)$.

Corollary 3. *The basic spectral counting function satisfies*

$$N_B(\lambda) := \#\{\lambda_j^B | \lambda_j^B \leq \lambda\} \sim \frac{\mathrm{Vol}(M'/\overline{\mathcal{F}})}{(4\pi)^{\overline{q}/2}\Gamma\left(\frac{\overline{q}}{2}+1\right)}\lambda^{\overline{q}/2}$$

as $\lambda \to \infty$.

For some specific types of Riemannian foliations (finite holonomy, codimension one or two), the heat asymptotics can be calculated explicitly.

Example:

If $(M', \mathcal{F})$ is codimension one without dense leaves, then

$$\text{tr}\left(e^{-t\Delta_B}\right) \sim \frac{1}{\sqrt{4\pi t}}\left(a_0 + a_1 t + \ldots\right),$$

where

$$a_0 = \text{Vol}(M'/\overline{\mathcal{F}}),$$
$$a_1 = \frac{3}{4}\left(\|\mathbf{H}\|_2\right)^2, \ldots$$

Thus, the spectrum of the basic Laplacian determines the transverse volume and the L^2 norm of the mean curvature of the foliation. This implies that one can "hear" if the foliation is minimal.