Lie Groups

Example *The orthogonal group $O(2)$ is the set of 2×2 matrices A such that $A^t A = I$. The special orthogonal group $SO(2)$ is the set of orthogonal matrices with determinant 1. Let's examine this. First, let's think of these matrices as a pair of column vectors. So if $A \in O(2)$, then*

$$A = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

with $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ as the two column vectors. So the equation $A^t A = I$ is equivalent to

$$\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} v_1^2 + v_2^2 & v_1 w_1 + v_2 w_2 \\ v_1 w_1 + v_2 w_2 & w_1^2 + w_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} v \cdot v & v \cdot w \\ v \cdot w & w \cdot w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $O(2)$ is the set of 2×2 matrices where the column vectors form an orthonormal frame, an orthonormal basis of $\mathbb{R}^2$. Note that $\begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}$ is the result after rotating the vector $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ clockwise 90 degrees, and the determinant of the matrix is

$$\det A = v_1 w_2 - v_2 w_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix},$$

which is 1 if (v, w) is an oriented frame (ie w is the result of a counterclockwise 90 degree rotation of v) and is -1 if (v, w) has the reverse orientation (ie w is the result of a clockwise 90 degree rotation of v). Thus $SO(2)$ corresponds to the set of oriented orthonormal frames.

Definition *A Lie group G is a manifold and group for which the multiplication map $\mu : G \times G \to G$ is smooth.*

Remark *It follows that the inverse map $i : G \to G$ defined by $i(g) = g^{-1}$ is also smooth. Proof: implicit fcn theorem + diagram*

$$\begin{array}{ccc} \mu^{-1}(e) \subseteq G \times G & \overset{\mu}{\to} & G \\ \pi_1 \downarrow & \nearrow & \\ G & & \end{array}$$

Example $(\mathbb{R}^n, +)$

Example S^1 *or* $T^n = S^1 \times \ldots \times S^1$

Example $Gl(n, \mathbb{F}) \subseteq \mathbb{F}^n$, *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$

Example $E_3 =$ *isometries of* $\mathbb{R}^3$ *(2 connected components) Let the orthogonal group* $O_3 < E_3$ *be the subgroup that fixes the origin, and let the special orthogonal group* $SO(3) = SO_3 < O_3$ *be the orientation-preserving elements of* O_3.

Visualizing $SO(3)$: Let u be a vector of length l in $\mathbb{R}^3$, corresponding to a rotation of angle l around the axis u. Redundancy: if $l = |u| = \pi$, u gives the same rotation as $-u$, so $SO(3)$ is the ball of radius π with antipodal points identified = $\mathbb{R}P^3$.

Matrix groups

Theorem *If* G *is a Lie group and* $H < G$, *then* H *is a Lie subgroup with the subspace topology if and only if* H *is closed.*

Example *Embed* $\mathbb{R}$ *as an irrational slope on* $\mathbb{R}^2 / \mathbb{Z}^2 = T^2$; *then this is a subgroup but is not a Lie subgroup.*

Note that

$$E_3 \cong \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \subseteq Gl(4, \mathbb{R}) \text{ such that } A \in O_3 \right\}$$

(b is the translation vector)

Classical Lie (sub)groups: $Sl(n, \mathbb{F})$ (det=1), $O(n)$ ($gg^t = \mathbf{1}$, orthogonal group), $SO(n)$ ($gg^t = \mathbf{1},$ det=1, special orthogonal group), $U(n)$ ($gg^* = \mathbf{1},$ unitary group), $SU(n)$ ($gg^* = \mathbf{1},$ det=1, special unitary group), $Sp(n) = \{g \in Gl(n, \mathbb{H}) : gg^* = \mathbf{1}\}$ (symplectic group).

Why study general Lie groups? Well, a standard group could be embedded in a funny way. For example, $\mathbb{R}$ can be embedded as (e^x) as matrices, or as $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ or as $\begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}$. Also, some examples are not matrix groups. For example, consider the following quotient of the Heisenberg group N: Let

$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathsf{R} \right\}$$

$$Z = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathsf{Z} \right\},$$

Let $G = N/Z$

These groups are important in quantum mechanics. Also, consider the following transformations of $L^2(\mathsf{R})$:

$$T_a(f)(x) = f(x-a)$$
$$M_b(f)(x) = e^{2\pi i b x} f(x)$$
$$U_c(f)(x) = e^{2\pi i c} f(x)$$

The group of operators of the form $T_a M_b U_c$ corresponds exactly to $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. In quantum mechanics, T_a corresponds to a unitary involution of momentum, and M is the momentum, U is phase.

Note that every Lie group is locally a matrix group.

Low dimensional, connected examples:

1. Dim 1: R, S^1
2. Dim 2: only nonabelian example is the space of affine transformations $x \mapsto mx + b$ of R.
3. Dim 3: SO_3, $SL_2(\mathsf{R})$, E_2, N (only new ones up to local isomorphism: G_1 and G_2 are locally isomorphic if there exist open neighborhoods around the identities that are homeomorphic through multiplication-preserving homeo)

Relationships between Lie groups

Observe that $U_2 = \{g : gg^* = \mathbf{1}\}$, $SU_2 = \{g \in U_2 : \det g = 1\}$.

For every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2$, then $g^* = g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$,

so

$$
\begin{aligned}
SU_2 &= \{g \in U_2 : \det g = 1\} \\
&= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} = S^3 \\
&= \left\{ \begin{pmatrix} t + ix & y + zi \\ -y + zi & t - ix \end{pmatrix} : (t, x, y, z) \in S^3 \right\} \\
&= \left\{ q = t\mathbf{1} + x\hat{\imath} + y\hat{\jmath} + z\hat{k} \in \mathbb{H} : (t, x, y, z) \in S^3 \right\} \\
&= Sp(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\imath} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\hat{\jmath} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\hat{k} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\end{aligned}
$$

satisfy the relations

$$
\begin{aligned}
&\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -\mathbf{1} \\
&\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k};\ \hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath} = \hat{\imath};\ \hat{k}\hat{\imath} = -\hat{\imath}\hat{k} = \hat{\jmath}.
\end{aligned}
$$