Introduction to Group Actions

Examples of sets of matrices that have the structure of groups ($A, B \in G$ implies $AB \in G$ and $A \in G$ implies $A^{-1} \in G$):

$O(3) = \{A \in M_3(\mathbb{R}) : A^T A = I\} = \{A \in M_3(\mathbb{R}) : A \text{ preserves geometry}\}.$

$$G_{2\times 2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in [0, 2\pi] \right\}$$

$SU(2) = \{A \in M_2(\mathbb{C}) : A^* A = I \text{ and } \det A = 1\}$

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \in M_2(\mathbb{C}) : |z| = 1 \right\}.$$

These sets are also smooth manifolds – we call them Lie groups. For example, you can think of $O(3)$ as a set inside $\mathbb{R}^9$ that satisfies a bunch of equations – a manifold of dimension 3, as it turns out. These groups act (multiplication with nice properties) on spaces:

$O(3)$ and subgroups $G_{2\times 2}$ and S^1 act on $\mathbb{R}^3$, S^2

$SU(2)$ and subgroup T act on $\mathbb{C}^2$, $\mathbb{R}^4$, S^3.

How do they "act"? By matrix multiplication. If G is the group that acts on the space X, then if $g, h \in G$ and $x \in X$, we have

$$g(hx) = (gh)x$$
$$Ix = x$$

An **orbit** of a point $x \in X$ is the set

$$\mathcal{O}_x = \{gx : g \in G\}.$$

For example, if $O(3)$ acts on S^2, then the orbit of any point is all of S^2. If S^1 acts on S^2 as above, then the orbit of a point is the latitude circle containing that point. Note that the north and south poles $(0,0,1)$ and $(0,0,-1)$ are fixed by all of S^1, so they are their own orbits. If G_{2x2} acts on S^2, then the orbit of a typical point will be the union of four points, although there are exceptions (for example, the orbit of $(-1,0,0)$ is itself, and the orbit of $(0,0,1)$ is $\{(0,0,1),(0,0,-1)\}$).

The **isotropy subgroup** or **stabilizer subgroup** G_x of a point $x \in X$ is the set

$$G_x = \{g : gx = x\}.$$

Let's look at some examples. Consider the north pole $NP = (0,0,1)$ on S^2 with the action of $O(3)$. Then if $A \in O(3)$ fixes $(0,0,1)$, that means that

$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

so the last column of A must be $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Multiplying both sides by A^T, we get

$$A^T A\begin{pmatrix}0\\0\\1\end{pmatrix} = A^T\begin{pmatrix}0\\0\\1\end{pmatrix},$$

so that

$$A^T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

also. This means that the last row of A is $\begin{pmatrix}0 & 0 & 1\end{pmatrix}$, so

$$A = \begin{pmatrix}a & b & 0\\c & d & 0\\0 & 0 & 1\end{pmatrix}.$$

Since $A \in O(3)$, this means that $\begin{pmatrix}a & b\\c & d\end{pmatrix} \in O(2)$ – the set of rotations and reflections in the (x,y) coordinates that fix the origin. So

$$O(3)_{NP} \cong O(2).$$

In general, the stabilizer $O(3)_{(x,y,z)}$ is a subgroup of $O(3)$ that is conjugate to the stabilizer at the NP, the group of rotations/reflections that fixes that one point.

Let's look at some other examples of isotropy subgroups. In the case of S^1 acting on S^2, the entire group fixes the north and south poles, and no element of the group fixes any other point. For $z \neq \pm 1,$

$$S^1_{(x,y,z)} = \{I\},$$

and

$$S^1_{(0,0,\pm 1)} = S^1.$$

Next, consider the group $G_{2\times 2}$ acting on S^2. The isotropy subgroup of each point in

$M_0 = \{(x,y,z) \in S^2 : z \neq 0, y \neq 0\}$ is $G_0 = \{(0,0)\}$, points of the set $M_2 = \{(x,y,z) \in S^2 : z = 0, y \neq 0\}$ have isotropy subgroup

$$G_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$, points in $M_1 = \{(x,y,z) \in S^2 : y = 0, z \neq 0\}$

have isotropy subgroup $G_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$, and points of $M_3 = \{(1,0,0), (-1,0,0)\}$ have isotropy subgroup $G_3 = G_{2\times 2}$.

An important theorem in transformation group theory is

Proposition *If M is a G-manifold, then for every $x \in M$, the orbit $\mathcal{O}_x$ is diffeomorphic to the homogeneous space G/G_x.*

For example, if we define $\phi : S^2 \to O(3)/S^1$ by

$$\phi(x,y,z) = gS^1,$$

where $g \in O(3)$ is any element of $O(3)$ that maps $NP = (0,0,1)$ to $(x,y,z) \in S^2$ (turns out it doesn't matter which one you pick), then this map is a diffeomorphism. The general situation is similar.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the *orbit type*. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K, and we say $[H] < [K]$ if $[H] \leq [K]$ and $[H] \neq [K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0], \ldots, [G_{r-1}]$ such that $[G_i] \leq [G_j]$ if and only if $i \leq j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other orbits are called *singular*. As a consequence, every isotropy subgroup H satisfies $[G_0] \leq [H]$. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the *stratum* corresponding to $[G_j]$. A stratum M_j is called a *most singular stratum* if there does not exist a stratum M_k such that $[G_j] < [G_k]$. It is known that each stratum is a G-invariant submanifold of M, and in fact a most singular stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $M_{\geq j} := \bigcup_{[G_k] \geq [G_j]} M_k$ is a closed, G-invariant submanifold. In the examples above, the notation is consistent to that in this paragraph. For example, the torus action on S^3 in Example ref: torusAction yields three strata, with M_0 being the principal stratum and M_1 and M_2 each being a most singular stratum.

If M_j is a most singular stratum, let $T_\varepsilon(M_j)$ denote an open tubular neighborhood of M_j of radius $\varepsilon > 0$. If ε is sufficiently small, then all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$.

Desingularization construction

With notation as in the previous section, we will construct a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the desingularization of M and is denoted $\widetilde{M}$. The significance of this construction is that it appears in the equivariant index theorem in cite: BKR, and the analysis of transversally elliptic operators on M may be replaced by analysis on $\widetilde{M}$, which is much easier to understand.

A sequence of constructions is used to construct N and $\widetilde{M} \subset N$. Let M_j be a most singular stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary. Since the $T_\varepsilon(M_j)$ is saturated (a union of G-orbits), the G-action lifts to N^1. Note that the strata of the G-action on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N^1_k = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G-manifold with one fewer stratum than M, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. In fact, N^1 is a branched double cover of M, branched over M_j. If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G-manifold N^1 to produce a new G-manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{\text{a most singular stratum}\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^{r-1}$ is a G-manifold with all orbits of type $[G_0]$ and is a 2^{r-1}-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^{r-1}$, which is a fundamental domain of M_0 in N.

As mentioned earlier, if M is equipped with a G-equivariant, transversally elliptic differential operator on sections of an equivariant vector bundle over M, then this data may be pulled back to the desingularization $\widetilde{M}$. Given the bundle and operator over N^j, simply form the invertible double of the operator on N^{j+1}, which is the double of the manifold with boundary $N^j \setminus T_\varepsilon(\Sigma)$, where Σ is a most singular stratum on N^j.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. Note that the isotropy type of all points of $\widetilde{M_{\geq j}}$ is $[G_j]$, and the $\widetilde{M_{\geq j}}/G$ is a smooth (open) manifold.

BKR J. Brüning, F. W. Kamber, and K. Richardson, *The eta invariant and equivariant index of transversally elliptic operators*, preprint in preparation.

PR I. Prokhorenkov and K. Richardson, *Natural equivariant Dirac operators*, preprint in preparation.