Generalized equivariant index theory

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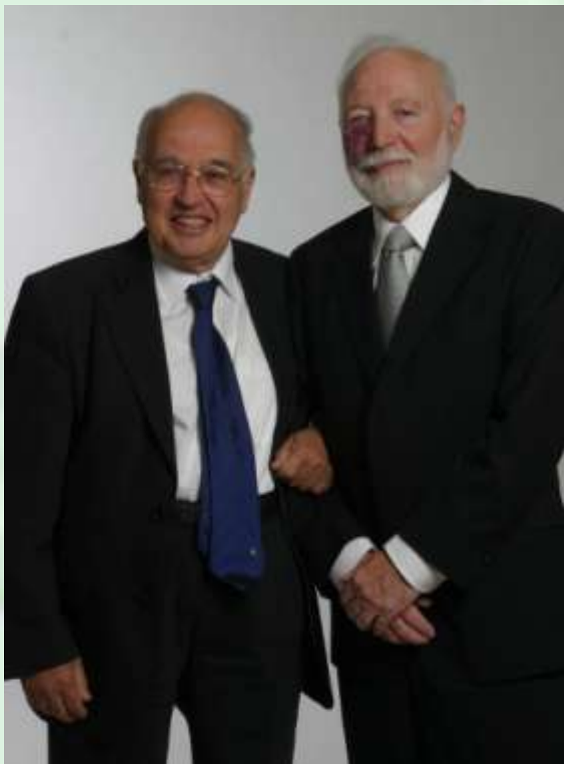
A long time ago
in a university
far, far away . . .

Index Theory and Non – Noncommutative geometry



Let M be a smooth, closed manifold, and let

$$D : \Gamma(M,E) \to \Gamma(M,F)$$

be an elliptic operator.

Index(D) := dim ker D – dim Diet Coke r D

= dim ker D – dim ker D^*

(in presence of metrics)

***Atiyah-Singer Index Thm*:**

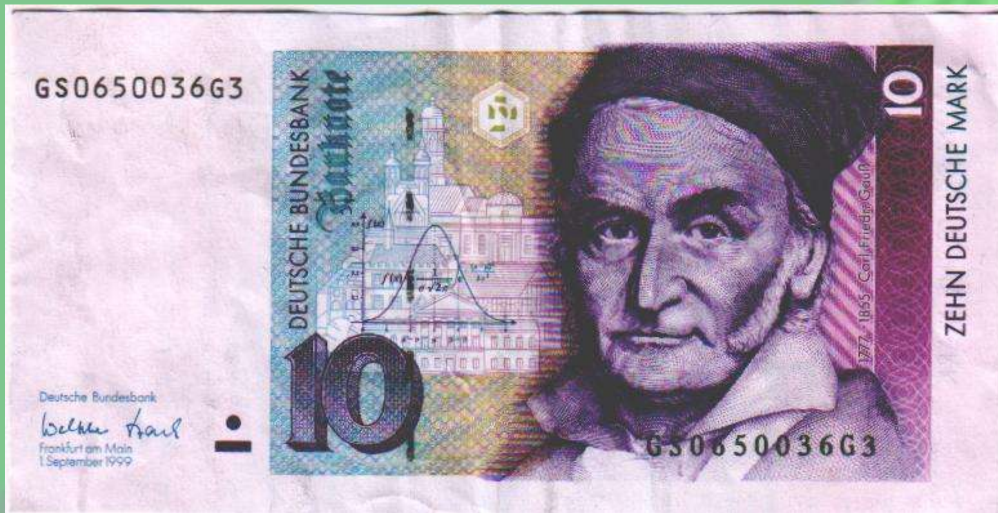
$$\text{Index}(D) = \int_M \text{AS}$$

Examples

$$\text{Index}\left(d+d^*\big|_{\text{even}\to\text{odd}}\right) = \chi(M) = \int_M K/_{2\pi}$$

$$\text{Index}\left(d+d^*\big|_{+\to-}\right) = \text{Sign}(M) = \int_M \text{Ł}$$

$$\text{Index}\left(\partial^+\right) = \int_M \hat{A}$$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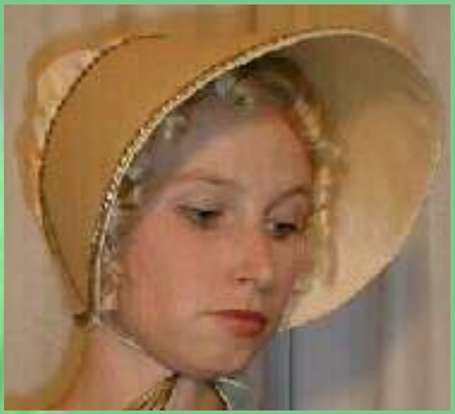


Equivariant Index Theory

Let G be a compact Lie group that acts by isometries on a closed, connected manifold M, and let $E = E^+ \oplus E^-$ be a graded *G-vector bundle* over M. Let $D: \Gamma(M, E^+) \to \Gamma(M, E^-)$ be *G-equivariant* and merely *transversally elliptic*.

The vector spaces ker D and ker D^* are typically infinite dimensional, and G acts as a group representation on each of them. These representations can be decomposed as direct sums of irreducible representations ρ. Thus, ker $D \cong \bigoplus m_\rho^+ [\rho]$ and ker $D^* \cong \bigoplus m_\rho^- [\rho]$, where $[\rho]$ is the equivalence class of the irreducible representation ρ, and $m_\rho^\pm$ is the multiplicity of that representation in the kernel. (Lemma: $m_\rho^\pm$ is finite.)

Equivariant Index Theory, continued

The *index multiplicity* of the representation ρ is defined to be

$$\mathrm{ind}_\rho(D) = m_\rho^+ - m_\rho^-$$

The index of D on the space of *invariant sections* is

$$\mathrm{ind}_1(D) = \mathrm{index}(D^G) = m_1^+ - m_1^-$$

The *representation-valued index* is the formal sum

$$\mathrm{ind}(D) = \bigoplus \left(m_\rho^+ - m_\rho^-\right)[\rho],$$

and the *distribution-valued index* is the virtual character

$$\mathrm{ind}_g(D) = \bigoplus \left(m_\rho^+ - m_\rho^-\right)\chi_\rho(g),$$

where $\chi_\rho = \mathrm{tr}(\rho)$ is the trace of the representation ρ.

Equivariant Index Theory Results

Formulas for $\mathrm{ind}_g (D)$:

Elliptic Case (Atiyah, Segal, Singer)

Transversally Elliptic Case (Berline and Vergne)

$$\mathrm{ind}_g (D)(\phi) = \int_{v \in T_e G} \left(\int_{M^{\exp(v)}} \mathrm{BV}_{(\phi(\exp(v)))} \right) dg_{(\exp(v))}$$

and

$$\mathrm{ind}_\rho (D) = ?????$$

- **partial results: if isotropy subgroups have constant dimension (Atiyah, Kawasaki – equivalent to orbifold index theorem)**

Index Theory on Manifolds with Boundary

Let X be a smooth, compact manifold with boundary Y, and let

$$D : \Gamma\left(X , E ; P_{\geq 0} \right) \rightarrow \Gamma\left(X , F ; P_{>0} \right)$$

be an elliptic operator.

Near Y, the operator D has the form

$$D = Z (\partial_r + A).$$

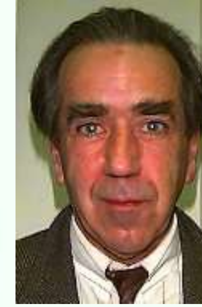
The boundary condition at Y is $P_{\geq 0}\, u = 0$, where $P_{\geq 0}$ is the spectral projection onto the eigenspaces of A with nonnegative eigenvalues.

***Atiyah-Patodi-Singer Index Thm*:**

$$\text{Index}(D \,;\, P_{\geq 0}) = \int_X \text{AS} \;-\; {}^{\eta}/_{2} \;-\; {}^{h}/_{2}$$

Basic Index Theory
on Riemannian foliations

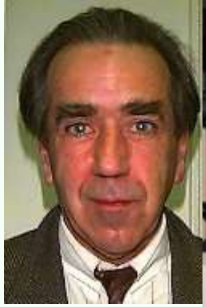
Joint work with Franz Kamber and Jochen Brüning



Let $(M,\mathcal{F})$ be a smooth, Riemannian foliation, and let ∇✎ be a basic connections on foliated bundles E✎ over M. Let $\Gamma_b(M,\mathcal{F};E$✎$) = \{\, u$ ♍ $\Gamma(M,E$✎$) \mid \nabla$✎$_X u = 0$ for every X ♍ $T\mathcal{F}\,\}$ be the set of *basic sections*. Let $D_b : \Gamma_b(M,\mathcal{F};E^+) \rightarrow \Gamma_b(M,\mathcal{F};E^-)$ be a *transversally elliptic operator* that preserves basic-ness.

Theorem (EK; R; BKR): (1) D_b is Fredholm.
(2) There is an explicit operator D on G-invariant sections of a G-equivariant vector bundle over the *basic manifold* such that $G=O(q)$ or $SO(q)$ and
$\mathrm{Index}(D_b) = \mathrm{Index}(D^G)$

The Invariant Index



Joint work with Franz Kamber and Jochen Brüning

As before, let G be a compact Lie group that acts by isometries on a closed, connected manifold M, and let $E = E^+ \oplus E^-$ be a graded G *-vector bundle* over M. Let $D: \Gamma(M, E^+) \to \Gamma(M, E^-)$ be G *-equivariant* and *transversally elliptic*.

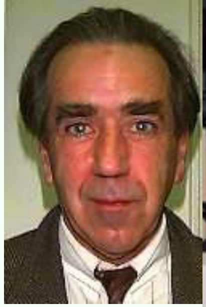
Problem: Compute

$$\mathrm{Index}(D^G) = \dim \ker D^G - \dim \ker (D^*)^G$$

<u>Background</u>: M is stratified by the action of G. The *isotropy subgroup* of $x \in M$ is the set of $g \in G$ that fix x. The *isotropy type* is the conjugacy class of the isotropy subgroup. The set of all $x \in M$ of a given isotropy type is called a *stratum*. The strata are foliated by orbits.

The Invariant Index

Joint work with Franz Kamber and Jochen Brüning




Assume that the action of G on M has only two strata, the principal stratum M_0 and the singular stratum Σ. Assume that $D = Z\,(\partial_r + r^{-1}D^S) + D^\Sigma = Z\,(\partial_r + r^{-1}D^S) * D^\Sigma$ near Σ. Assume D^S does not depend on r and that its invariant eigenvalues do not depend on Σ; assume that D^Σ is the pullback of an operator on Σ. And ...

Main Theorem:

$$\text{Index}\,[D^G] = \int_{M_0/G} (AS)^G - \tfrac{1}{2}\eta\left[(D^S)^G\right]\cdot \text{Index}\left[(D^\Sigma)^G\right]$$

Riemannian Foliation version of Gauss-Bonnet Theorem

$$\chi(M, \mathcal{F}) = \sum_i \chi(L_i, \mathcal{F}, \mathcal{O}_i)\, \chi(M(H_i)/\mathcal{F}_c),$$

where $M(H_i)$ is the closure of the stratum corresponding to infinitessimal holonomy type $[H_i]$, L_i is a representative leaf closure of type $[H_i]$, $\mathcal{F}_c$ is the singular leaf closure foliation, and $\mathcal{O}_i$ is the orientation line bundle of $M(H_i)/\mathcal{F}_c$.