Desingularizing Compact Lie Group Actions

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This talk includes joint work with Jochen Brüning, Franz Kamber, and Igor Prokhorenkov – various papers.

Plan of the talk

- Again, Gomapsumida to the actual organizers
- Manifolds and compact Lie group actions
- Stratifications of *G*-manifolds
- Desingularization construction
- Equivariant index theorem
- Natural equivariant Dirac operators
- Further comments

Lie group actions: G-manifolds

- G – compact Lie group that acts smoothly on
- M – smooth, connected, closed manifold
- *Assume that the action is effective*
  (ie $\{g \in G : gx = x \text{ for all } x \in M\} = \emptyset$ *)*
- *Choose a Riemannian metric for which G acts by isometries.*
- **Orbit:** $O_x = \{gx : g \in G\}$
- **Isotropy subgroup:** $G_x < G$

  $$G_x = \{g \in G : gx = x\}$$

Isotropy subgroups along an orbit

- We have $G_{gx} = gG_xg^{-1}$
- Thus, **the conjugacy class of the isotropy subgroup is fixed along the orbit**.
- The conjugacy class $[G_x] = \{gG_xg^{-1} \, : g \in G\}$ is called the **orbit type** of the point or orbit.
- On any G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types.

Partial order and stratification

- Given subgroups $H, K < G$ that occur as isotropy subgroups on M, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K, and we say $[H] < [K]$ if $[H] \leq [K]$ and $[H] \neq [K]$.
- Enumerate $[G_0], \ldots, [G_{r-1}]$ such that $[G_i] \leq [G_j]$ iff $i \leq j$.
- Define the **stratum**

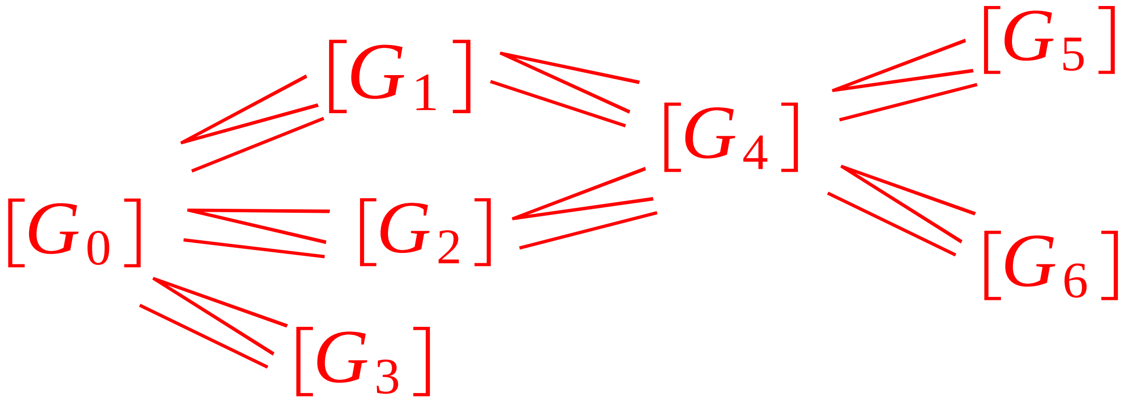
$$M_j = \{x \in M : [G_x] = [G_j]\}$$

G-manifold stratification

- M_0 is the principal stratum, which is open and dense in M. The isotropy subgroup class corresponding to this stratum is the smallest and is less than every other stratum. This is called the set of regular points, and all other strata are called **singular strata**.
- Each stratum M_j is a G-invariant submanifold of M.

Stratification, continued

- Typical partial ordering may look like this:



- In the above case, the strata M_3, M_5, and M_6 are called **most singular strata**; the corresponding isotropy subgroups are contained in no larger subgroups. These strata are always closed submanifolds.

Examples

- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $M = S^2 \subset \mathbb{R}^3$ by

$$(1,0)(x,y,z) = (x,-y,z),\ (0,1)(x,y,z) = (x,y,-z)$$

- $G = S^1$ acts on $M = S^2 \subset \mathbb{R}^3$ by

$$e^{i\theta}(x,y,z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), z)$$

- $G = S^1 \times S^1$ acts on $M = S^3 \subset \mathbb{R}^4$ by

$$(e^{i\theta}, e^{i\alpha})(x,y,z,w) = (\ x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta),$$

$$z\cos(\alpha) - w\sin(\alpha), z\sin(\alpha) + w\cos(\alpha)\ )$$

Tubular neighborhood of a stratum

- If M_j is a most singular stratum, let $T_\varepsilon(M_j)$ denote an open tubular neighborhood of M_j radius $\varepsilon > 0$. If ε is sufficiently small, then each orbit in $T_\varepsilon(M_j) \setminus M_j$ is of some type $[G_k]$, where $[G_k] < [G_j]$.
- Let $M_{\geq j} = \bigcup_{[G_k] \geq [G_j]} M_k$

  Then this G-invariant submanifold is also closed.

Desingularization construction

1. Let M_j be a most singular stratum.
   Let $T_\varepsilon(M_j)$ denote an open tubular neighborhood, as above.

2. Let
   $$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

3. Let $\widetilde{M}^1 = M \setminus T_\varepsilon(M_j)$, a fundamental domain of $M \setminus M_j$ in N^1.

4. Repeat with M replaced by N^1 to get N^2, and $\widetilde{M}^2 = \widetilde{M}^1 \setminus \{\text{a most singular stratum}\}$

5. The end result is $\widetilde{M} = \widetilde{M}^{r-1}$ after $r-1$ steps.

Application: Equivariant Index Theorem

Theorem (Brüning, Kamber, R):

$$\mathrm{ind}^{\rho}(D) = \int_{\widetilde{M}/G} A_0^{\rho}(x)\, \widetilde{|dx|} + \sum_{j,a,b} C_{jab} \int_{\widetilde{M_{\geq j}}/G} \left(-\eta\left(D_j^{S+,\sigma_a}\right) + h\left(D_j^{S+,\sigma_a}\right)\right) A^{\rho_0}_{j,\sigma_b^*}(x)\, \widetilde{|dx|}.$$

Here, $D = \left\{ Z_j \left(\nabla^E_{\partial_r} + \frac{1}{r} D_j^S \right) \right\} * D^{M_{\geq j}}$

Natural Equivariant Dirac operators

- Let $F_O \overset{p}{\to} M$ be the principal $O(n)$-bundle of orthonormal frames.
- The G-action on M induces the differential action on the bundle of frames, for which the isotropy groups are all trivial.
- The resulting G-orbits on F_O form a Riemannian fiber bundle (ie wrt Sasakian metric), $F_O \overset{\pi}{\to} F_O / G$, and the base is a compact $O(n)$-manifold.

Equivariant vector bundles

- Let $E \to F_O$ be a Hermitian vector bundle that is equivariant wrt the $G \times O(n)$ -action.
- Let $\rho : G \to U(V_\rho)$ and $\sigma : O(n) \to U(W_\sigma)$ be irreducible unitary representations.
- Define the bundles $\mathsf{E}^\sigma \to M$ and $\mathsf{T}^\rho \to F_O/G$ by

$$\mathsf{E}_x^\sigma = \Gamma(p^{-1}(x), E)^\sigma,$$

$$\mathsf{T}_y^\rho = \Gamma(\pi^{-1}(y), E)^\rho$$

Isomorphisms of subspaces of sections

Theorem (Prokhorenkov, R):

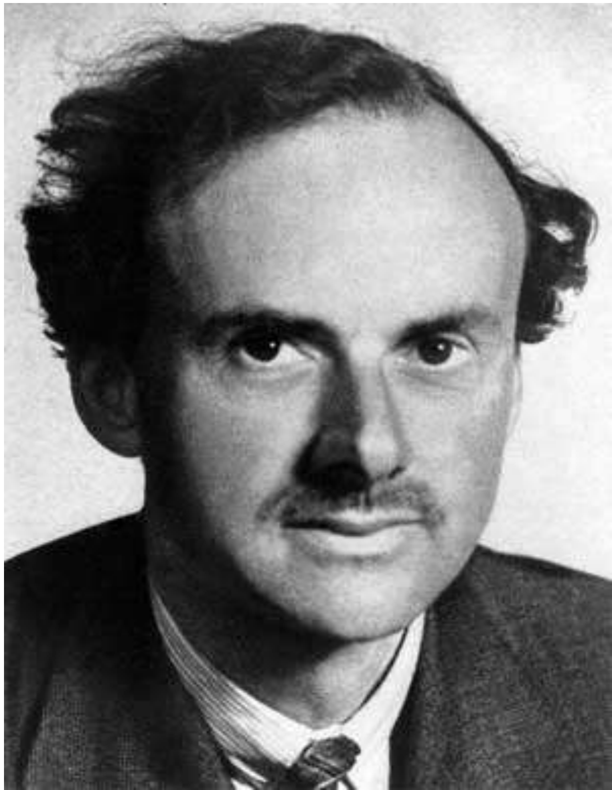
For any irreducible representations ρ, σ as above, there is an explicit isomorphism

$$\Gamma(M, \mathsf{E}^{\sigma})^{\rho} \to \Gamma(F_O / G, \mathsf{T}^{\rho})^{\sigma}$$

that extends to an L^2-isometry.

Equivariant Dirac operators

On F_O, the foliation by G-orbits is a Riemannian foliation, and thus we may define a basic Dirac operator there !



Recall: Basic Dirac operators

- Given a Riemannian foliation **(*M,F*)** of codim ***q*** with compatible bundle-like metric, ***E*** a foliated vector bundle,

$$D_{\mathrm{tr}}\, s = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i}^{E} s\,,$$

$$D_b s = \frac{1}{2}(D_{\mathrm{tr}} + D_{\mathrm{tr}}^{*})s = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i}^{E} s - \frac{1}{2}\kappa_b^{\sharp} \cdot s$$

- References: Glazebrook-Kamber, S.D.Jung

New Equivariant Dirac operators

- We define operators

$$D_M^\sigma : \Gamma(M, \mathsf{E}^\sigma) \to \Gamma(M, \mathsf{E}^\sigma)$$

$$D_{F_O/G}^\rho : \Gamma(F_O/G, \mathsf{T}^\rho) \to \Gamma(F_O/G, \mathsf{T}^\rho)$$

by restricting

$$D_b : \Gamma_b(F_O, E) \to \Gamma_b(F_O, E)$$

to subspaces of sections.

More subspaces of sections

- Let $\alpha : G \to U(V_\alpha),\ \beta : G \to U(W_\beta)$ be irreducible representations; let

$$(D_M^\sigma)^\alpha : \Gamma(M, \mathsf{E}^\sigma)^\alpha \to \Gamma(M, \mathsf{E}^\sigma)^\alpha$$

be the restriction of D_M^σ, and define

$$(D_{F_O/G}^\rho)^\beta : \Gamma(F_O/G, \mathsf{T}^\rho)^\beta \to \Gamma(F_O/G, \mathsf{T}^\rho)^\beta$$

similarly.

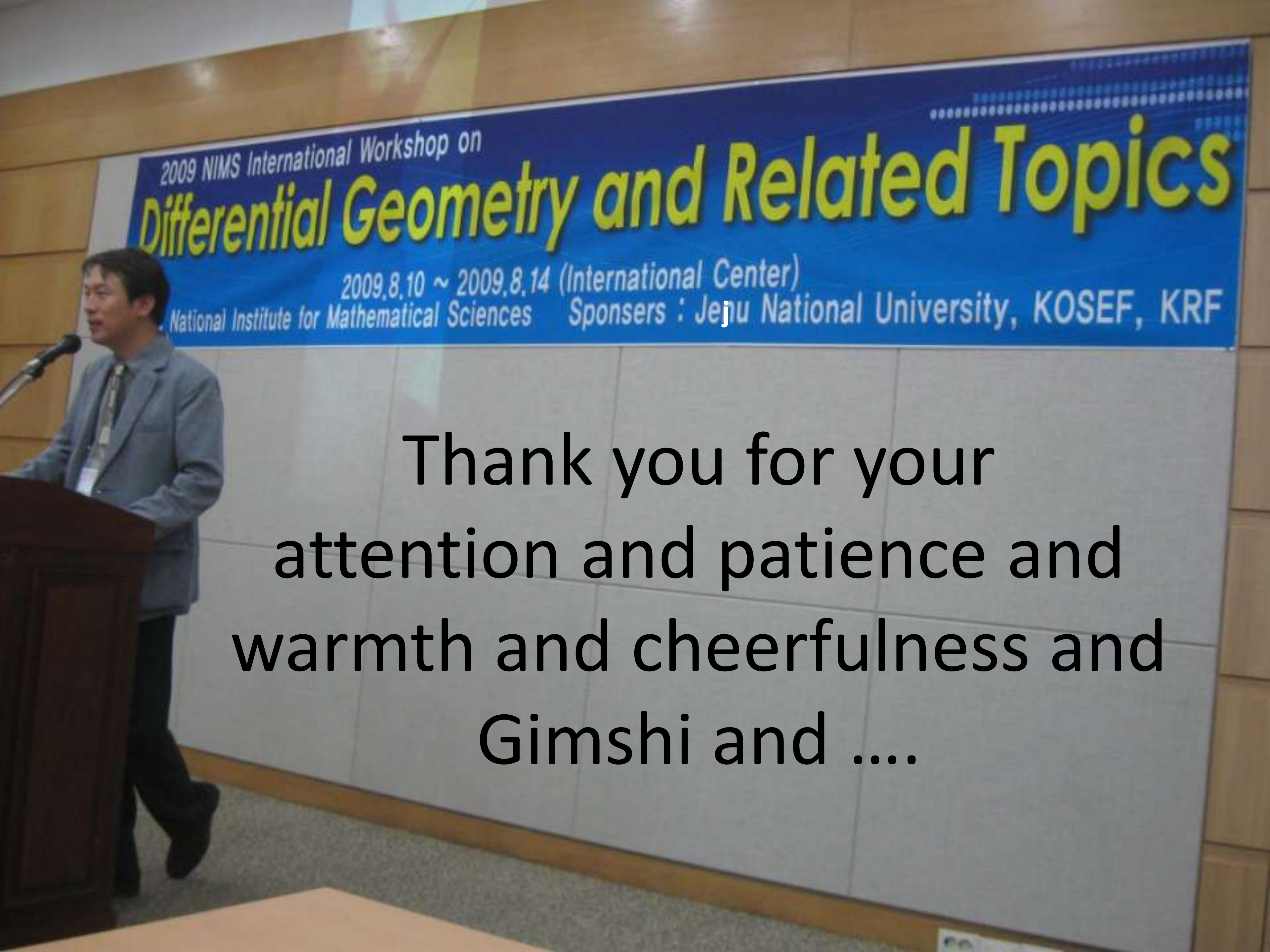
Transverse Ellipticity and Spectrum

Proposition (Prokhorenkov, R):

The operator D_M^σ is transversally elliptic and G-equivariant, and $D_{F_O/G}^\rho$ is elliptic and $O(n)$-equivariant, and the closures of these operators are self-adjoint. The operators $(D_M^\sigma)^\rho$ and $(D_{F_O/G}^\rho)^\sigma$ have identical discrete spectrum, and the corresponding eigenspaces are conjugate via Hilbert space isomorphisms.

Further remarks

- It turns out that questions about the transversally elliptic operators D_M^σ can be reduced to questions about the elliptic operators $D^\rho_{F_O/G}$.
- The operators D_M^σ play the same role for equivariant analysis as the standard Dirac operators do in index theory and analysis of elliptic operators on closed manifolds.



Thank you for your attention and patience and warmth and cheerfulness and Gimshi and ….