TRANSVERSAL DIRAC OPERATORS ON DISTRIBUTIONS, FOLIATIONS, AND G-MANIFOLDS LECTURE NOTES

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ABSTRACT. In these lectures, we investigate generalizations of the ordinary Dirac operator to manifolds with additional structure. In particular, if the manifold comes equipped with a distribution and an associated Clifford algebra action on a bundle over the manifold, one may define a transversal Dirac operator associated to this structure. We investigate the geometric and analytic properties of these operators, and we apply the analysis to the settings of Riemannian foliations and of manifolds endowed with Lie group actions. Among other results, we show that although a bundle-like metric on the manifold is needed to define the basic Dirac operator on a Riemannian foliation, its spectrum depends only on the Riemannian foliation structure. Using these ideas, we produce a type of basic cohomology that satisfies Poincaré duality on transversally oriented Riemannian foliations. Also, we show that there is an Atiyah-Singer type theorem for the equivariant index of operators that are transversally elliptic with respect to a compact Lie group action. This formula relies heavily on the stratification of the manifold with group action and contains eta invariants and curvature forms. These notes contain exercises at the end of each subsection and are meant to be accessible to graduate students.

1. INTRODUCTION TO ORDINARY DIRAC OPERATORS

1.1. **The Laplacian.** The **Laplace operator** (or simply, **Laplacian**) is the famous differential operator Δ on $\mathbb{R}^n$ defined by

$$\Delta h = -\sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2}, h \in C^{\infty}(\mathbb{R}^n)$$

The solutions to the equation $\Delta h = 0$ are the **harmonic functions**. This operator is present in both the heat equation and wave equations of physics.

$$\text{Heat equation:} \quad \frac{\partial u(t,x)}{\partial t} + \Delta_x u(t,x) = 0$$

$$\text{Wave equation} \quad : \quad \frac{\partial^2 u(t,x)}{\partial t^2} + \Delta_x u(t,x) = 0$$

The sign of the Laplacian is chosen so that it is a nonnegative operator. If $\langle u, v \rangle$ denotes the L^2 inner product on complex-valued functions on $\mathbb{R}^n$, by integrating by parts, we see that

$$\begin{aligned} \langle \Delta u, u \rangle &= \int_{\mathbb{R}^n} (\Delta u) \overline{u} \\ &= \int_{\mathbb{R}^n} |\nabla u|^2 \end{aligned}$$

if u is compactly supported, where $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right)$ is the gradient vector. The calculation verifies the nonnegativity of Δ.

The same result holds if instead the Laplace operator acts on the space of smooth functions on a closed Riemannian manifold (compact, without boundary); the differential operator is modified in a natural way to account for the metric. That is, if the manifold is isometrically embedded in Euclidean space, the Laplacian of a function on that manifold agrees with the Euclidean Laplacian above if that function is extended to be constant in the normal direction in a neighborhood of the embedded submanifold. One may also define the Laplacian on differential forms in precisely the same way; the Euclidean Laplacian on forms satisfies

$$\Delta \left[u\left(x\right) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_p} \right] = \left(\Delta u\right)\left(x\right) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_p}.$$

These standard formulas for the Laplace operator suffice if the Riemannian manifold is flat (for example, flat tori), but it is convenient to give a coordinate-free description for this operator. If (M, g) is a smooth manifold with metric $g = (\bullet, \bullet)$, the volume form on M satisfies $\mathrm{dvol} = \sqrt{\det g}\, dx$. The metric induces an isomorphism $v_p \to v_p^\flat$ between vectors and one forms at $p \in M$, given by

$$v_p^\flat \left(w_p\right) := \left(v_p, w_p\right), \ w_p \in T_p M.$$

Thus, given an orthonormal basis $\{e_j : 1 \leq j \leq n\}$ of the tangent space $T_p M$, we declare the corresponding dual basis $\left\{ e_j^\flat : 1 \leq j \leq n \right\}$ to be orthonormal, and in general we declare $\left\{ e_\alpha^\flat = e_{\alpha_1}^\flat \wedge ... \wedge e_{\alpha_r}^\flat \right\}_{|\alpha|=k}$ to be an orthonormal basis of r-forms at a point. Then the L^2 inner product of r-forms on M is defined by

$$\langle \gamma, \beta \rangle = \int_M \left(\gamma, \beta\right) \ \mathrm{dvol} \ .$$

Next, if $d : \Omega^r\left(M\right) \to \Omega^{r+1}\left(M\right)$ is the exterior derivative on smooth r-forms, we define $\delta : \Omega^{r+1}\left(M\right) \to \Omega^r\left(M\right)$ to be the formal adjoint of d with respect to the L^2 inner product. That is, if $\omega \in \Omega^{r+1}\left(M\right)$, we define $\delta \omega$ by requiring

$$\langle \gamma, \delta \omega \rangle = \langle d\gamma, \omega \rangle$$

for all $\gamma \in \Omega^r\left(M\right)$. Then the **Laplacian on differential r-forms on** M is defined to be

$$\Delta = \delta d + d\delta : \Omega^r\left(M\right) \to \Omega^r\left(M\right).$$

It can be shown that Δ is an essentially self-adjoint operator. The word *essentially* means that the space of smooth forms needs to be closed with respect to a certain Hilbert space norm, called a Sobolev norm.

We mention that in many applications, vector-valued Laplacians and Laplacians on sections of vector bundles are used.

Exercise 1. *Explicitly compute the formal adjoint δ for d restricted to compactly supported forms in Euclidean space, and verify that the $\delta d + d\delta$ agrees with the Euclidean Laplacian on r-forms.*

Exercise 2. *Show that a smooth r-form $\alpha \in \Omega^r\left(M\right)$ is harmonic, meaning that $\Delta \alpha = 0$, if and only if $d\alpha$ and $\delta\alpha$ are both zero.*

Exercise 3. *Explicitly compute the set of harmonic r-forms on the 2-dimensional flat torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Verify the **Hodge Theorem** in this specific case; that is, show that the space of harmonic r-forms is isomorphic to the r-dimensional de Rham cohomology group $H^r\left(M\right)$.*

Exercise 4. *Suppose that α is a representative of a cohomology class in $H^r(M)$. Show that α is a harmonic form if and only if α is the element of the cohomology class with minimum L^2 norm.*

Exercise 5. *If (g_{ij}) is the local matrix for the metric with $g_{ij} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, show that the matrix (g^{ij}) defined by $g^{ij} = (dx_i, dx_j)$ is the inverse of the matrix (g_{ij}).*

Exercise 6. *If $\alpha = \sum_{j=1}^{n} \alpha_j(x)\, dx_j$ is a one-form on a Riemannian manifold of dimension n, where $g^{ij} = (dx_i, dx_j)$ is the local metric matrix for one-forms, verify that the formal adjoint δ satisfies*

$$\delta(\alpha) = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g}\, \alpha_j\right).$$

Exercise 7. *Show that if $f \in C^\infty(M)$, then*

$$\int_M \Delta f\ \mathrm{dvol} = 0.$$

1.2. **The ordinary Dirac operator.** The original motivation for constructing a Dirac operator was the need of a first-order differential operator whose square is the Laplacian. Dirac needed such an operator in order to make some version of quantum mechanics that is compatible with special relativity. Specifically, suppose that $D = \sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j}$ is a first-order, constant-coefficient differential operator on $\mathbb{R}^n$ such that D^2 is the ordinary Laplacian on $\mathbb{R}^n$. Then one is quickly led to the equations

$$\begin{aligned} c_i^2 &= -1, \\ c_i c_j + c_j c_i &= 0,\ i \neq j \end{aligned}$$

Clearly, this is impossible if we require each $c_j \in \mathbb{C}$. However, if we allow matrix coefficients, we are able to find such matrices; they are called **Clifford matrices**. In the particular case of $\mathbb{R}^3$, we may use the famous **Pauli spin matrices**

$$c_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ c_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ c_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The vector space $\mathbb{C}^k$ on which the matrices and derivatives act is called the vector space of spinors. It can be shown that the minimum dimension k satisfies $k = 2^{\lfloor n/2 \rfloor}$. The matrices can be used to form an associated **Clifford multiplication of vectors**, written $c(v)$, defined by

$$c(v) = \sum_{j=1}^{n} v_j c_j\ ,$$

where $v = (v_1, ..., v_n)$. Note that $c : \mathbb{R}^n \to \mathrm{End}\left(\mathbb{C}^k\right)$ such that

$$c(v)\, c(w) + c(w)\, c(v) = -2\,(v, w),\ v, w \in \mathbb{R}^n.$$

If M is a closed Riemannian manifold, we desire to find a Hermitian vector bundle $E \to M$ and a first-order differential operator $D : \Gamma(E) \to \Gamma(E)$ on sections of E such that its square is a Laplacian plus a lower-order differential operator. This implies that each E_x is a $\mathbb{C}\mathrm{l}(T_x M)$-module, where $\mathbb{C}\mathrm{l}(T_x M)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(E_x)$ generated by a Clifford

multiplication of tangent vectors. Then the Dirac operator associated to the Clifford module E is defined for a local orthonormal frame $(e_j)_{j=1}^n$ of TM to be

$$D = \sum_{j=1}^n c(e_j) \nabla_{e_j},$$

where c denotes Clifford multiplication and where ∇ is a metric connection on E satisfying the compatibility condition

$$\nabla_V (c(W) s) = c(\nabla_V W) s + c(W) \nabla_V s$$

for all sections $s \in \Gamma(E)$ and vector fields $V, W \in \Gamma(TM)$. We also require that Clifford multiplication of vectors is skew-adjoint with respect to the L^2 inner product, meaning that

$$\langle c(v) s_1, s_2 \rangle = -\langle s_1, c(v) s_2 \rangle$$

for all $v \in \Gamma(TM)$, $s_1, s_2 \in \Gamma(E)$. It can be shown that the expression for D above is independent of the choice of orthonormal frame of TM. In the case where E has the minimum possible rank $k = 2^{\lfloor n/2 \rfloor}$, we call E a **complex spinor bundle** and D a **spin^c Dirac operator**. If such a bundle exists over a smooth manifold M, we say that M is spin^c. There is a mild topological obstruction to the existence of such a structure; the third integral Stiefel-Whitney class of TM must vanish.

Often the bundle E comes equipped with a grading $E = E^+ \oplus E^-$ such that D maps $\Gamma(E^+)$ to $\Gamma(E^-)$ and vice-versa. In these cases, we often restrict our attention to $D : \Gamma(E^+) \to \Gamma(E^-)$

Examples of ordinary Dirac operators are as follows:

- The **de Rham operator** is defined to be

$$d + \delta : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$$

  from even forms to odd forms. In this case, the Clifford multiplication is given by $c(v) = v^\flat \wedge -i(v)$, where $v \in T_x M$ and $i(v)$ denotes interior product, and ∇ is the ordinary Levi-Civita connection extended to forms.
- If M is even-dimensional, the **signature operator** is defined to be

$$d + \delta : \Omega^+(M) \to \Omega^-(M)$$

  from **self-dual** to **anti-self-dual** forms. This grading is defined as follows. Let $*$ denote the **Hodge star operator** on forms, defined as the unique endomorphism of the bundle of forms such that $* : \Omega^r(M) \to \Omega^{n-r}(M)$ and

$$\alpha \wedge *\beta = (\alpha, \beta) \, \text{dvol}, \ \alpha, \beta \in \Omega^r(M).$$

  Then observe that the operator

$$\bigstar = i^{r(r-1)+\frac{n}{2}} * : \Omega^r(M) \to \Omega^{n-r}(M)$$

  satisfies $\bigstar^2 = 1$. Then it can be shown that $d + \delta$ anticommutes with $\bigstar$ and thus maps the $+1$ eigenspace of $\bigstar$, denoted $\Omega^+(M)$, to the -1 eigenspace of $\bigstar$, denoted $\Omega^-(M)$. Even though the bundles have changed from the previous example, the expression for Clifford multiplication is the same.

- If M is complex, then the **Dolbeault operator** is defined to be

$$\overline{\partial} + \overline{\partial}^* : \Omega^{0,\text{even}}(M) \to \Omega^{0,\text{odd}}(M),$$

  where the differential forms involve wedge products of $d\overline{z_j}$ and the differential $\overline{\partial}$ differentiates only with respect to the $\overline{z_j}$ variables.
- The **spin^c Dirac operator** has already been mentioned above. The key point is that the vector bundle $S \to M$ in this case has the minimum possible dimension. When M is even-dimensional, then the spinor bundle decomposes as $S^+ \oplus S^-$, and $D : \Gamma(S^+) \to \Gamma(S^-)$. The spinor bundle S is unique up to tensoring with a complex line bundle.

For more information on Dirac operators, spin manifolds, and Clifford algebras, we refer the reader to [38] and [53]. Often the operators described above are called *Dirac-type operators*, with the word "Dirac operator" reserved for the special examples of the spin or spin^c Dirac operator. Elements of the kernel of a spin or spin^c Dirac operator are called **harmonic spinors**.

Exercise 8. *Let the Dirac operator D on the two-dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ be defined using c_1 and c_2 of the Pauli spin matrices. Find a decomposition of the bundle as $S^+ \oplus S^-$, and calculate $\ker(D|_{S^+})$ and $\ker(D|_{S^-})$. Find all the eigenvalues and corresponding eigensections of $D^+ = D|_{S^+}$.*

Exercise 9. *On an n-dimensional manifold M, show that $*^2 = (-1)^{r(n-r)}$ and $\bigstar^2 = 1$ when restricted to r-forms.*

Exercise 10. *On $\mathbb{R}^4$ with metric $ds^2 = dx_1^2 + 4dx_2^2 + dx_3^2 + (1 + \exp(x_1))^2 dx_4^2$, let $\omega = x_1^2 x_2 dx_2 \wedge dx_4$. Find $*\omega$ and $\bigstar\omega$.*

Exercise 11. *Calculate the signature operator on T^2, and identify the subspaces $\Omega^+(T^2)$ and $\Omega^-(T^2)$.*

Exercise 12. *Show that $-i\frac{\partial}{\partial\theta}$ is a Dirac operator on $S^1 = \left\{ e^{i\theta} : \theta \in \mathbb{R} \right\}$. Find all the eigenvalues and eigenfunctions of this operator.*

Exercise 13. *Show that if S and T are two anticommuting linear transformations from a vector space V to itself, then if E_λ is the eigenspace of S corresponding to an eigenvalue λ, then TE_λ is the eigenspace of S corresponding to the eigenvalue $-\lambda$.*

Exercise 14. *Show that if δ^r is the adjoint of $d : \Omega^{r-1}(M) \to \Omega^r(M)$, then*

$$\delta^r = (-1)^{nr+n+1} * d*$$

on $\Omega^r(M)$.

Exercise 15. *Show that if the dimension of M is even and δ^r is the adjoint of $d : \Omega^{r-1}(M) \to \Omega^r(M)$, then*

$$\delta^r = -\bigstar d\bigstar$$

on $\Omega^r(M)$. Is this true if the dimension is odd?

Exercise 16. *Show that $d + \delta$ maps $\Omega^+(M)$ to $\Omega^-(M)$.*

Exercise 17. *Show that if we write the Dirac operator for* $\mathbb{R}^3$

$$D = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} + c_3 \frac{\partial}{\partial x_3}$$

using the Pauli spin matrices in geodesic polar coordinates

$$D = Z \left(\frac{\partial}{\partial r} + D^S \right),$$

then ZD^S *restricts to a* spin^c *Dirac operator on the unit sphere* S^2*, and* Z *is Clifford multiplication by the vector* $\frac{\partial}{\partial r}$.

Exercise 18. *Show that* $d + \delta = \sum_{j=1}^n c(e_j) \nabla_{e_j}$*, with the definition of Clifford multiplication given in the notes.*

Exercise 19. *Show that the expression* $\sum_{j=1}^n c(e_j) \nabla_{e_j}$ *for the Dirac operator is independent of the choice of orthonormal frame.*

1.3. **Properties of Dirac operators.** Here we describe some very important properties of Dirac operators.

First, Dirac operators are *elliptic*. Both the Laplacian and Dirac operators are examples of such operators. Very roughly, the word *elliptic* means that the operators differentiate in all possible directions. To state more precisely what this means, we need to discuss what is called the *principal symbol* of a differential (or pseudodifferential) operator.

Very roughly, the principal symbol is the set of matrix-valued leading order coefficients of the operator. If $E \to M$, $F \to M$ are two vector bundles and $P : \Gamma(E) \to \Gamma(F)$ is a differential operator of order k acting on sections, then in local coordinates of a local trivialization of the vector bundles, we may write

$$P = \sum_{|\alpha|=k} s_\alpha(x) \frac{\partial^k}{\partial x^\alpha} + \text{lower order terms},$$

where the sum is over all possible multi-indices $\alpha = (\alpha_1, ..., \alpha_k)$ of length $|\alpha| = k$, and each $s_\alpha(x) \in \text{Hom}(E_x, F_x)$ is a linear transformation. If $\xi = \sum \xi_j dx_j \in T_x^* M$ is a nonzero covector at x, we define the **principal symbol** $\sigma(P)(\xi)$ of P at ξ to be

$$\sigma(P)(\xi) = i^k \sum_{|\alpha|=k} s_\alpha(x) \xi^\alpha \in \text{Hom}(E_x, F_x),$$

with $\xi^\alpha = \xi_{\alpha_1} \xi_{\alpha_2} ... \xi_{\alpha_k}$ (some people leave out the i^k). It turns out that by defining it this way, it is invariant under coordinate transformations. One coordinate-free definition of $\sigma(P)_x : T_x^*(M) \to \text{Hom}(E_x, F_x)$ is as follows. For any $\xi \in T_x^*(M)$, choose a locally-defined function f such that $df_x = \xi$. Then we define the operator

$$\sigma_m(P)(\xi) = \lim_{t \to \infty} \frac{1}{t^m} \left(e^{-itf} P e^{itf} \right),$$

where $\left(e^{-itf} P e^{itf} \right)(u) = e^{-itf} \left(P \left(e^{itf} u \right) \right)$. Then the order k of the operator and principal symbol are defined to be

$$
\begin{aligned}
k &= \sup \{ m : \sigma_m(P)(\xi) < \infty \} \\
\sigma(P)(\xi) &= \sigma_k(P)(\xi) \ .
\end{aligned}
$$

With this definition, the principal symbol of any differential (or even pseudodifferential) operator can be found. Pseudodifferential operators are more general operators that can be defined locally using the Fourier transform and include such operators as the square root of the Laplacian.

An **elliptic differential (or pseudodifferential) operator** P on M is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is invertible for all nonzero $\xi \in T^*M$.

From the exercises at the end of this section, we see that the symbol of any Dirac operator $D = \sum c(e_j) \nabla_{e_j}$ is

$$\sigma(D)(\xi) = ic\left(\xi^{\#}\right),$$

and the symbol of the associated Laplacian D^2 is

$$\sigma\left(D^2\right) = \left(ic\left(\xi^{\#}\right)\right)^2 = \|\xi\|^2,$$

which is clearly invertible for $\xi \neq 0$. Therefore both D and D^2 are elliptic.

We say that an operator P is **strongly elliptic** if there exists $c > 0$ such that

$$\sigma(D)(\xi) \geq c\,|\xi|^2$$

for all nonzero $\xi \in T^*M$. The Laplacian and D^2 are strongly elliptic.

Following are important properties of elliptic operators P, which now apply to Dirac operators and their associated Laplacians:

- **Elliptic regularity**: if the coefficients of P are smooth, then if Pu is smooth, then u is smooth. As a consequence, if the order of P is greater than zero, then the kernel and all other eigenspaces of P consist of smooth sections.
- Elliptic operators are Fredholm when the correct Sobolev spaces of sections are used.
- Ellipticity implies that the spectrum of P consists of eigenvalues. Strong ellipticity implies that the spectrum is discrete and has the only limit point at infinity. In particular, the eigenspaces are finite-dimensional and consist of smooth sections. This now applies to any Dirac operator, because its square is strongly elliptic.
- If P is a second order elliptic differential operator with no zero$^{\text{th}}$ order terms, strong ellipticity implies the maximum principle for the operator P.
- Many inequalities for elliptic operators follow, like Gårding's inequality, the elliptic estimates, etc.

See [57], [53], and [54] for more information on elliptic differential and pseudodifferential operators on manifolds.

Next, any Dirac operator $D : \Gamma(E) \to \Gamma(E)$ is **formally self-adjoint**, meaning that when restricted to smooth compactly-supported sections $u, v \in \Gamma(E)$ it satisfies

$$\langle Du, v \rangle = \langle u, Dv \rangle.$$

Since D is elliptic and if M is closed, it then follows that D is essentially self-adjoint, meaning that there is a Hilbert space $H^1(E)$ such that $\Gamma(E) \subset H^1(E) \subset L^2(E)$ such that the closure of D in $H^1(E)$ is a truly self-adjoint operator defined on the whole space. In this particular case, $H^1(E)$ is an example of a Sobolev space, which is the closure of $\Gamma(E)$ with respect to the norm $\|u\|_1 = \|u\| + \|Du\|$, where $\|\bullet\|$ denotes the ordinary L^2-norm.

We now show the proof that D is formally self-adjoint. If the local bundle inner product on E is $(\bullet, \bullet)$, we have

$$
\begin{aligned}
(Du, v) &= \sum \left(c(e_j) \nabla_{e_j} u, v \right) = \sum - \left(\nabla_{e_j} u, c(e_j) v \right) \\
&= \sum -e_j \left(u, c(e_j) v \right) + \left(u, \nabla_{e_j} \left(c(e_j) v \right) \right),
\end{aligned}
$$

since $c(e_j)$ is skew-adjoint and ∇ is a metric connection. Using the compatibility of the connection, we have

$$
(Du, v) = \sum -e_j \left(u, c(e_j) v \right) + \left(u, c \left(\nabla_{e_j} e_j \right) v \right) + \left(u, c(e_j) \nabla_{e_j} v \right).
$$

Next, we use the fact that we are allowed to choose the local orthonormal frame in any way we wish. If we are evaluating this local inner product at a point $x \in M$, we choose the orthonormal frame (e_i) so that all covariant derivatives of e_i vanish at x. Now, the middle term above vanishes, and

$$
(Du, v) = (u, Dv) + \sum -e_j \left(u, c(e_j) v \right).
$$

Next, if ω denotes the one-form defined by $\omega(X) = (u, c(X) v)$ for $X \in \Gamma(TM)$, then an exercise at the end of this section implies that

$$
(\delta \omega)(x) = \left(\sum -e_j \left(u, c(e_j) v \right) \right)(x),
$$

with our choice of orthonormal frame. Hence,

$$
(Du, v) = (u, Dv) + \delta \omega,
$$

which is a general formula now valid at all points of M. After integrating over M we have

$$
\begin{aligned}
\langle Du, v \rangle &= \langle u, Dv \rangle + \int_M \delta \omega \, \mathrm{dvol} \\
&= \langle u, Dv \rangle + \int_M \left(d(1), \omega \right) \, \mathrm{dvol} \\
&= \langle u, Dv \rangle.
\end{aligned}
$$

Thus, D is formally self-adjoint.

Exercise 20. *Find the principal symbol of the wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ on $\mathbb{R}^2$, and determine if it is elliptic.*

Exercise 21. *If P_1 and P_2 are two differential operators such that the composition $P_1 P_2$ is defined, show that*

$$
\sigma(P_1 P_2)(\xi) = \sigma(P_1)(\xi) \sigma(P_2)(\xi).
$$

Exercise 22. *Prove that if $D = \sum c(e_j) \nabla_{e_j}$ is a Dirac operator, then*

$$
\sigma(D)(\xi) = ic\left(\xi^{\#}\right) \quad \text{and} \quad \sigma\left(D^2\right)(\xi) = \|\xi\|^2
$$

for all $\xi \in T^* M$. Use the coordinate-free definition.

Exercise 23. *Show that if ω is a one-form on M, then*

$$
(\delta \omega)(x) = - \left(\sum_{j=1}^n e_j \left(\omega(e_j) \right) \right)(x),
$$

if $(e_1, ..., e_n)$ is a local orthonormal frame of TM chosen so that

$$\left(\nabla_{e_j} e_k\right)(x) = 0$$

at $x \in M$, for every $j, k \in \{1, ..., n\}$.

1.4. **The Atiyah-Singer Index Theorem.** Given Banach spaces S and T, a bounded linear operator $L : S \rightarrow T$ is called **Fredholm** if its range is closed and its kernel and cokernel $T / L(S)$ are finite dimensional. The **index** of such an operator is defined to be

$$\mathrm{ind}(L) = \dim \ker(L) - \dim \mathrm{coker}(L),$$

and this index is constant on continuous families of such L. In the case where S and T are Hilbert spaces, this is the same as

$$\mathrm{ind}(L) = \dim \ker(L) - \dim \ker(L^*).$$

The index determines the connected component of L in the space of Fredholm operators. We will be specifically interested in this integer in the case where L is a Dirac operator.

For the case of the de Rham operator, we have

$$\ker(d + \delta) = \ker(d + \delta)^2 = \ker \Delta,$$

so that

$$\mathcal{H}^r(M) = \ker\left((d + \delta)|_{\Omega^r}\right)$$

is the space of harmonic forms of degree r, which by the Hodge theorem is isomorphic to $H^r(M)$, the rth de Rham cohomology group. Therefore, index

$$\begin{aligned}
\mathrm{ind}\left((d + \delta)|_{\Omega^{\mathrm{even}}}\right) \quad : \quad & = \dim \ker\left((d + \delta)|_{\Omega^{\mathrm{even}}}\right) - \dim \ker\left((d + \delta)|_{\Omega^{\mathrm{even}}}\right)^* \\
& = \dim \ker\left((d + \delta)|_{\Omega^{\mathrm{even}}}\right) - \dim \ker\left((d + \delta)|_{\Omega^{\mathrm{odd}}}\right) \\
& = \chi(M),
\end{aligned}$$

the Euler characteristic of M.

In general, suppose that D is an elliptic operator of order m on sections of a vector bundle $E^{\pm}$ over a smooth, compact manifold M. Let $H^s(\Gamma(M, E^{\pm}))$ denote the Sobolev s-norm completion of the space of sections $\Gamma(M, E)$, with respect to a chosen metric. Then D can be extended to be a bounded linear operator $\overline{D_s} : H^s(\Gamma(M, E^+)) \rightarrow H^{s-m}(\Gamma(M, E^-))$ that is Fredholm, and $\mathrm{ind}(D) := \mathrm{ind}(\overline{D_s})$ is well-defined and independent of s. In the 1960s, the researchers M. F. Atiyah and I. Singer proved that the index of an elliptic operator on sections of a vector bundle over a smooth manifold satisfies the following formula ([5], [6]):

$$\begin{aligned}
\mathrm{ind}(D) & = \int_M \mathrm{ch}(\sigma(D)) \wedge \mathrm{Todd}(T_{\mathbb{C}} M) \\
& = \int_M \alpha(x) \, \mathrm{dvol}(x),
\end{aligned}$$

where $\mathrm{ch}(\sigma(D))$ is a form representing the Chern character of the principal symbol $\sigma(D)$, and $\mathrm{Todd}(T_{\mathbb{C}} M)$ is a form representing the Todd class of the complexified tangent bundle $T_{\mathbb{C}} M$; these forms are characteristic forms derived from the theory of characteristic classes and depend on geometric and topological data. The local expression for the relevant term of the integrand, which is a multiple of the volume form $\mathrm{dvol}(x)$, can be written in terms of curvature and the principal symbol and is denoted $\alpha(x) \, \mathrm{dvol}(x)$.

Typical examples of this theorem are some classic theorems in global analysis. As in the earlier example, let $D = d + \delta$ from the space of even forms to the space of odd forms

on the manifold M of dimension n, where as before δ denotes the L^2-adjoint of the exterior derivative d. Then the elements of $\ker(d+\delta)$ are the even harmonic forms, and the elements of the cokernel can be identified with odd harmonic forms. Moreover,

$$
\begin{aligned}
\operatorname{ind}(d+\delta) &= \dim H^{\mathrm{even}}(M) - \dim H^{\mathrm{odd}}(M) \\
&= \chi(M), \text{ and} \\
\int_M \operatorname{ch}(\sigma(d+\delta)) \wedge \operatorname{Todd}(T_{\mathbb{C}}M) &= \frac{1}{(2\pi)^n} \int_M \operatorname{Pf},
\end{aligned}
$$

where Pf is the Pfaffian, which is, suitably interpreted, a characteristic form obtained using the square root of the determinant of the curvature matrix. In the case of 2-manifolds ($n = 2$), Pf is the Gauss curvature times the area form. Thus, in this case the Atiyah-Singer Index Theorem yields the generalized Gauss-Bonnet Theorem.

Another example is the operator $D = d + d^*$ on forms on an even-dimensional manifold, this time mapping the self-dual to the anti-self-dual forms. This time the Atiyah-Singer Index Theorem yields the equation (called the Hirzebruch Signature Theorem)

$$
\operatorname{Sign}(M) = \int_M L,
$$

where $\operatorname{Sign}(M)$ is signature of the manifold, and L is the Hirzebruch L-polynomial applied to the Pontryagin forms.

If a manifold is spin, then the index of the spin Dirac operator is the $\widehat{A}$ genus ("A-roof" genus) of the manifold. Note that the spin Dirac operator is an example of a spin^c Dirac operator where the spinor bundle is associated to a principal $\operatorname{Spin}(n)$ bundle. Such a structure exists when the second Stiefel-Whitney class is zero, a stronger condition than the spin^c condition. The $\widehat{A}$ genus is normally a rational number but must agree with the index when the manifold is spin.

Different examples of operators yield other classical theorems, such as the Hirzebruch-Riemann-Roch Theorem, which uses the Dolbeault operator.

All of the first order differential operators mentioned above are examples of Dirac operators. If M is spin^c, then the Atiyah-Singer Index Theorem reduces to a calculation of the index of Dirac operators (twisted by a bundle). Because of this and the Thom isomorphism in K-theory, the Dirac operators and their symbols play a very important role in proofs of the Atiyah-Singer Index Theorem. For more information, see [6], [38].

Exercise 24. *Prove that if $L : \mathcal{H}_1 \to \mathcal{H}_2$ is a Fredholm operator between Hilbert spaces, then $\operatorname{coker}(L) \cong \ker(L^*)$.*

Exercise 25. *Suppose that $P : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator, and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is an orthogonal decomposition. If P maps $\mathcal{H}^+$ into $\mathcal{H}^-$ and vice versa, prove that the adjoint of the restriction $P : \mathcal{H}^+ \to \mathcal{H}^-$ is the restriction of P to $\mathcal{H}^-$. Also, find the adjoint of the operator $P' : \mathcal{H}^+ \to \mathcal{H}$ defined by $P'(h) = P(h)$.*

Exercise 26. *If D is an elliptic operator and E_λ is an eigenspace of D^*D corresponding to the eigenvalue $\lambda \neq 0$, then $D(E_\lambda)$ is the eigenspace of DD^* corresponding to the eigenvalue λ. Conclude that the eigenspaces of D^*D and DD^* corresponding to nonzero eigenvalues have the same (finite) dimension.*

Exercise 27. *Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth function, and let $L : \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator with discrete spectrum. Let $P_\lambda : \mathcal{H} \to E_\lambda$ be the orthogonal projection to the*

eigenspace corresponding to the eigenvalue λ. *We define the operator* $f(L)$ *to be*

$$f(L) = \sum_\lambda f(\lambda) P_\lambda,$$

assuming the right hand side converges. Assuming $f(L)$, $g(L)$, *and* $f(L)g(L)$ *converge, prove that* $f(L)g(L) = g(L)f(L)$. *Also, find the conditions on a function* f *such that* $f(L) = L$.

Exercise 28. *Show that if* P *is a self-adjoint Fredholm operator, then*

$$\mathrm{ind}(D) = \mathrm{tr}(\exp(-tD^*D)) - \mathrm{tr}(\exp(-tDD^*))$$

for all $t > 0$, *assuming that* $\exp(-tD^*D)$, $\exp(-tDD^*)$, *and their traces converge.*

Exercise 29. *Find all homeomorphism types of surfaces* S *such that a metric* g *on* S *has Gauss curvature* K_g *that satisfies* $-5 \leq K_g \leq 0$ *and volume that satisfies* $1 \leq Vol_g(S) \leq 4$.

Exercise 30. *Find an example of a smooth closed manifold* M *such that every possible metric on* M *must have nonzero* L *(the Hirzebruch* L*-polynomial applied to the Pontryagin forms).*

Exercise 31. *Suppose that on a certain manifold the* $\widehat{A}$*-genus is* $\frac{3}{4}$. *What does this imply about Stiefel-Whitney classes?*

2. Transversal Dirac operators on distributions

This section contains some of the results in [45], joint work with I. Prokhorenkov.

The main point of this section is to provide some ways to analyze operators that are not elliptic but behave in some ways like elliptic operators on sections that behave nicely with respect to a designated *transverse subbundle* $Q \subseteq TM$. A **transversally elliptic differential (or pseudodifferential) operator** P on M with respect to the transverse distribution $Q \subseteq TM$ is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is required to be invertible only for all nonzero $\xi \in Q^* \subseteq T^*M$. In later sections, we will be looking at operators that are transversally elliptic with respect to the orbits of a group action, and in this case Q is the normal bundle to the orbits, which may have different dimensions at different points of the manifold. In this section, we will restrict to the case where Q has constant rank.

Now, let $Q \subset TM$ be a smooth distribution, meaning that $Q \to M$ is a smooth subbundle of the tangent bundle. Assume that a $\mathbb{C}l(Q)$-module structure on a complex Hermitian vector bundle E is given, and we will now define transverse Dirac operators on sections of E. Similar to the above, M is a closed Riemannian manifold, $c : Q \to \mathrm{End}(E)$ is the Clifford multiplication on E, and ∇^E is a $\mathbb{C}l(Q)$ connection that is compatible with the metric on M; that is, Clifford multiplication by each vector is skew-Hermitian, and we require

$$\nabla_X^E(c(V)s) = c\left(\nabla_X^Q V\right)s + c(V)\nabla_X^E s$$

for all $X \in \Gamma(TM)$, $V \in \Gamma Q$, and $s \in \Gamma E$.

Remark 2.1. *For a given distribution* $Q \subset TM$, *it is always possible to obtain a bundle of* $\mathbb{C}l(Q)$*-modules with Clifford connection from a bundle of* $\mathbb{C}l(TM)$*-Clifford modules, but not all such* $\mathbb{C}l(Q)$ *connections are of that type.*

Let $L = Q^\perp$, let $(f_1, ..., f_q)$ be a local orthonormal frame for Q, and let $\pi : TM \to Q$ be the orthogonal projection. We define the Dirac operator A_Q corresponding to the distribution Q as

$$A_Q = \sum_{j=1}^{q} c(f_j) \nabla_{f_j}^E. \tag{2.1}$$

This definition is again independent of the choice of orthonormal frame; in fact it is the composition of the maps

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{\cong} \Gamma(TM \otimes E) \xrightarrow{\pi} \Gamma(Q \otimes E) \xrightarrow{c} \Gamma(E).$$

We now calculate the formal adjoint of A_Q, in precisely the same way that we showed the formal self-adjointness of the ordinary Dirac operator. Letting (s_1, s_2) denote the pointwise inner product of sections of E, we have

$$
\begin{aligned}
(A_Q s_1, s_2) &= \sum_{j=1}^{q} \left(c(\pi f_j) \nabla_{f_j}^E s_1, s_2 \right) \\
&= \sum_{j=1}^{q} - \left(\nabla_{f_j}^E s_1, c(\pi f_j) s_2 \right).
\end{aligned}
$$

Since ∇^E is a metric connection,

$$
\begin{aligned}
(A_Q s_1, s_2) &= \sum \left(-f_j (s_1, c(\pi f_j) s_2) + \left(s_1, \nabla_{f_j}^E c(\pi f_j) s_2 \right) \right) \\
&= \sum \Bigg(-f_j (s_1, c(\pi f_j) s_2) + \left(s_1, c(\pi f_j) \nabla_{f_j}^E s_2 \right) \\
&\quad + \left(s_1, c\left(\pi \nabla_{f_j}^M \pi f_j \right) s_2 \right) \Bigg),
\end{aligned}
\tag{2.2}
$$

by the $\mathbb{Cl}(Q)$-compatibility. Now, we do not have the freedom to choose the frame so that the covariant derivatives vanish at a certain point, because we know nothing about the distribution Q. Hence we define the vector fields

$$V = \sum_{j=1}^{q} \pi \nabla_{f_j}^M f_j \ , \ \ H^L = \sum_{j=q+1}^{n} \pi \nabla_{f_j}^M f_j \ .$$

Note that H^L is precisely the mean curvature of the distribution $L = Q^\perp$. Further, letting ω be the one-form defined by

$$\omega(X) = (s_1, c(\pi X) s_2),$$

and letting $(f_1, ..., f_q, f_{q+1}, ..., f_n)$ be an extension of the frame of Q to be an orthonormal frame of TM,

$$
\begin{aligned}
\delta\omega &= -\sum_{j=1}^{n} i\left(f_j\right) \nabla_{f_j} \omega \\
&= -\sum_{j=1}^{n} \left(f_j \omega\left(f_j\right) - \omega\left(\nabla_{f_j} f_j\right)\right) \\
&= \sum_{j=1}^{n} \left(-f_j\left(s_1, c\left(\pi f_j\right) s_2\right) + \left(s_1, c\left(\pi \nabla_{f_j}^M f_j\right) s_2\right)\right) \\
&= \left(s_1, c\left(V + H^L\right) s_2\right) + \sum_{j=1}^{n} \left(-f_j\left(s_1, c\left(\pi f_j\right) s_2\right)\right) \\
&= \left(s_1, c\left(V + H^L\right) s_2\right) + \sum_{j=1}^{q} \left(-f_j\left(s_1, c\left(\pi f_j\right) s_2\right)\right).
\end{aligned}
$$

From (2.2) we have

$$
\begin{aligned}
\left(A_Q s_1, s_2\right) &= \delta\omega - \left(s_1, c\left(V + H^L\right) s_2\right) \\
&\quad + \left(s_1, A_Q s_2\right) + \left(s_1, c\left(V\right) s_2\right) \\
\left(A_Q s_1, s_2\right) &= \delta\omega + \left(s_1, A_Q s_2\right) - \left(s_1, c\left(H^L\right) s_2\right).
\end{aligned}
$$

Thus, by integrating over the manifold (which sends $\delta\omega$ to zero), we see that the formal L^2-adjoint of A_Q is

$$
A_Q^* = A_Q - c\left(H^L\right).
$$

Since $c\left(H^L\right)$ is skew-adjoint, the new operator

$$
D_Q = A_Q - \frac{1}{2} c\left(H^L\right) \tag{2.3}
$$

is formally self-adjoint.

A quick look at [15] yields the following.

Theorem 2.2. (in [45]) *For each distribution $Q \subset TM$ and every bundle E of $\mathrm{Cl}\left(Q\right)$-modules, the transversally elliptic operator D_Q defined by (2.1) and (2.3) is essentially self-adjoint.*

Remark 2.3. *In general, the spectrum of D_Q is not necessarily discrete. In the case of Riemannian foliations, we identify Q with the normal bundle of the foliation, and one typically restricts to the space of basic sections. In this case, the spectrum of D_Q restricted to the basic sections is discrete.*

Exercise 32. *Let $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, and consider the distribution Q defined by the vectors parallel to $(1, r)$ with $r \in \mathbb{R}$. Calculate the operator D_Q and its spectrum, where the Clifford multiplication is just complex number multiplication (on a trivial bundle $E_x = \mathbb{C}$). Does it make a difference if r is rational?*

Exercise 33. *With M and Q as in the last exercise, let E be the bundle $\wedge^* Q^*$. Now calculate D_Q and its spectrum.*

Exercise 34. *Consider the radially symmetric Heisenberg distribution, defined as follows. Let $\alpha \in \Omega^1(\mathbb{R}^3)$ be the differential form*

$$
\begin{aligned}
\alpha &= dz - \frac{1}{2}r^2 d\theta \\
&= dz - \frac{1}{2}\left(xdy - ydx\right).
\end{aligned}
$$

Note that

$$d\alpha = -rdr \wedge d\theta = -dx \wedge dy,$$

so that α is a contact form because

$$\alpha \wedge d\alpha = -dx \wedge dy \wedge dz \neq 0$$

at each point of H. The two-dimensional distribution $Q \subset \mathbb{R}^3$ is defined as $Q = \ker \alpha$. Calculate the operator D_Q.

Exercise 35. *Let (M, α) be a manifold of dimension $2n + 1$ with contact form α; that is, α is a one-form such that*

$$\alpha \wedge (d\alpha)^n$$

is everywhere nonsingular. The distribution $Q = \ker \alpha$ is the contact distribution. Calculate the mean curvature of Q in terms of α.

Exercise 36. *(This example is in the paper [45].) We consider the torus $M = (\mathbb{R}/2\pi\mathbb{Z})^2$ with the metric $e^{2g(y)}dx^2 + dy^2$ for some 2π-periodic smooth function g. Consider the orthogonal distributions $L = \operatorname{span}\{\partial_y\}$ and $Q = \operatorname{span}\{\partial_x\}$. Let E be the trivial complex line bundle over M, and let $\mathbb{Cl}(Q)$ and $\mathbb{Cl}(L)$ both act on E via $c(\partial_y) = i = c\left(e^{-g(y)}\partial_x\right)$. Show that the mean curvatures of these distributions are*

$$H^Q = -g'(y)\,\partial_y \ \text{and}\ H^L = 0$$

From formulas (2.1) and (2.3),

$$A_L = i\partial_y, \ \text{and}\ D_L = i\left(\partial_y + \frac{1}{2}g'(y)\right).$$

Show that the spectrum $\sigma(D_L) = \mathbb{Z}$ is a set consisting of eigenvalues of infinite multiplicity.

Exercise 37. *In the last example, show that the operator*

$$D_Q = ie^{-g(y)}\partial_x$$

has spectrum

$$\sigma(D_Q) = \bigcup_{n \in \mathbb{Z}} n\,[a, b],$$

where $[a, b]$ is the range of $e^{-g(y)}$.

3. Basic Dirac operators on Riemannian foliations

The results of this section are joint work with G. Habib and can be found in [25] and [26].

3.1. **Invariance of the spectrum of basic Dirac operators.** Suppose a closed manifold M is endowed with the structure of a Riemannian foliation $(M, \mathcal{F}, g_Q)$. The word **Riemannian** means that there is a metric on the local space of leaves — a holonomy-invariant transverse metric g_Q on the normal bundle $Q = TM/T\mathcal{F}$. The phrase **holonomy-invariant** means the transverse Lie derivative $\mathcal{L}_X g_Q$ is zero for all leafwise vector fields $X \in \Gamma(T\mathcal{F})$.

We often assume that the manifold is endowed with the additional structure of a **bundle-like metric** [47], i.e. the metric g on M induces the metric on $Q \simeq N\mathcal{F} = (T\mathcal{F})^\perp$. Every Riemannian foliation admits bundle-like metrics that are compatible with a given $(M, \mathcal{F}, g_Q)$ structure. There are many choices, since one may freely choose the metric along the leaves and also the transverse subbundle $N\mathcal{F}$. We note that a bundle-like metric on a smooth foliation is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant. There are topological restrictions to the existence of bundle-like metrics (and thus Riemannian foliations). Important examples of requirements for the existence of a Riemannian foliations may be found in [35], [31], [43], [56], [58], [55]. One geometric requirement is that, for any metric on the manifold, the orthogonal projection

$$P : L^2\left(\Omega\left(M\right)\right) \to L^2\left(\Omega\left(M, \mathcal{F}\right)\right)$$

must map the subspace of smooth forms onto the subspace of smooth basic forms ([44]). Recall that **basic forms** are forms that depend only on the transverse variables. The space $\Omega(M, \mathcal{F})$ of basic forms is defined invariantly as

$$\Omega\left(M, \mathcal{F}\right) = \left\{\beta \in \Omega\left(M\right) : i\left(X\right)\beta = 0 \text{ and } i\left(X\right)d\beta = 0 \text{ for all } X \in \Gamma\left(T\mathcal{F}\right)\right\}.$$

The basic forms $\Omega(M, \mathcal{F})$ are preserved by the exterior derivative, and the resulting cohomology is called **basic cohomology** $H^*(M, \mathcal{F})$ (see also Section). It is known that the basic cohomology groups are finite-dimensional in the Riemannian foliation case. See [19], [20], [35], [32], [33], [21] for facts about basic cohomology and Riemannian foliations. For later use, the **basic Euler characteristic** is defined to be

$$\chi\left(M, \mathcal{F}\right) = \sum\left(-1\right)^j \dim H^j\left(M, \mathcal{F}\right).$$

We now discuss the construction of the basic Dirac operator, a construction which requires a choice of bundle-like metric. See [32], [33], [35], [18], [22], [45], [28], [29], [24], [25], [12], [13] for related results. Let $(M, \mathcal{F})$ be a Riemannian manifold endowed with a Riemannian foliation. Let $E \to M$ be a foliated vector bundle (see [31]) that is a bundle of $\mathbb{C}l(Q)$ Clifford modules with compatible connection ∇^E. This means that foliation lifts to a horizontal foliation in TE. Another way of saying this is that connection is flat along the leaves of $\mathcal{F}$. When this happens, it is always possible to choose a basic connection for E — that is, a connection for which the connection and curvature forms are actually (Lie algebra-valued) basic forms.

Let $A_{N\mathcal{F}}$ and $D_{N\mathcal{F}}$ be the associated transversal Dirac operators as in the previous section. The transversal Dirac operator $A_{N\mathcal{F}}$ fixes the basic sections $\Gamma_b(E) \subset \Gamma(E)$ (i.e. $\Gamma_b(E) = \{s \in \Gamma(E) : \nabla^E_X s = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\}$) but is not symmetric on this subspace. Let $P_b : L^2\left(\Gamma\left(E\right)\right) \to L^2\left(\Gamma_b\left(E\right)\right)$ be the orthogonal projection, which can be shown to map smooth sections to smooth basic sections. We define the basic Dirac operator to be

$$\begin{aligned} D_b \quad &: \quad = P_b D_{N\mathcal{F}} P_b \\ &= \quad A_{N\mathcal{F}} - \frac{1}{2}c\left(\kappa_b^\sharp\right) : \Gamma_b\left(E\right) \to \Gamma_b\left(E\right) \end{aligned}$$

$$P_b A_{N\mathcal{F}} P_b = A_{N\mathcal{F}} P_b, \ \ P_b c\left(\kappa^\sharp\right) P_b = c\left(\kappa_b^\sharp\right) P_b$$

Here, κ_b is the L^2-orthogonal projection of κ onto the space of basic forms as explained above, and $\kappa_b^\sharp$ is the corresponding basic vector field. Then D_b is an essentially self-adjoint, transversally elliptic operator on $\Gamma_b(E)$. The local formula for D_b is

$$D_b s = \sum_{i=1}^{q} e_i \cdot \nabla^E_{e_i} s - \frac{1}{2}\kappa_b^\sharp \cdot s \ ,$$

where $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q. Then D_b has discrete spectrum ([22], [18], [13]).

An example of the basic Dirac operator is as follows. Using the bundle $\wedge^* Q^*$ as the Clifford bundle with Clifford action $e\cdot \ = e^* \wedge -e^* \lrcorner$ in analogy to the ordinary de Rham operator, we have

$$\begin{aligned} D_b \ &= \ d + \delta_b - \frac{1}{2}\kappa_b \lrcorner - \frac{1}{2}\kappa_b \wedge . \\ &= \ \widetilde{d} + \widetilde{\delta}. \end{aligned}$$

One might have incorrectly guessed that $d + \delta_b$ is the basic de Rham operator in analogy to the ordinary de Rham operator, for this operator is essentially self-adjoint, and the associated basic Laplacian yields basic Hodge theory that can be used to compute the basic cohomology. The square D_b^2 of this operator and the basic Laplacian Δ_b do have the same principal transverse symbol. In [25], we showed the invariance of the spectrum of D_b with respect to a change of metric on M in any way that keeps the transverse metric on the normal bundle intact (this includes modifying the subbundle $N\mathcal{F} \subset TM$, as one must do in order to make the mean curvature basic, for example). That is,

Theorem 3.1. *(In* [25] *) Let $(M,\mathcal{F})$ be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E \to M$. The spectrum of the basic Dirac operator is the same for every possible choice of bundle-like metric that is associated to the transverse metric on the quotient bundle Q.*

We emphasize that the basic Dirac operator D_b depends on the choice of bundle-like metric, not merely on the Clifford structure and Riemannian foliation structure, since both projections $T^*M \to Q^*$ and P depend on the leafwise metric. It is well-known that the eigenvalues of the basic Laplacian Δ_b (closely related to D_b^2) depend on the choice of bundle-like metric; for example, in [52, Corollary 3.8], it is shown that the spectrum of the basic Laplacian on functions determines the L^2-norm of the mean curvature on a transversally oriented foliation of codimension one. If the foliation were taut, then a bundle-like metric could be chosen so that the mean curvature is identically zero, and other metrics could be chosen where the mean curvature is nonzero. This is one reason why the invariance of the spectrum of the basic Dirac operator is a surprise.

Exercise 38. *Suppose that $\mathcal{S}$ is a closed subspace of a Hilbert space $\mathcal{H}$, and let $L : \mathcal{H} \to \mathcal{H}$ be a bounded linear map such that $L(\mathcal{S}) \subseteq \mathcal{S}$. Let $L_\mathcal{S}$ denote the restriction $L_\mathcal{S} : \mathcal{S} \to \mathcal{S}$ defined by $L_\mathcal{S}(v) = L(v)$ for all $v \in \mathcal{S}$. Prove that the adjoint of $L_\mathcal{S}$ satisfies $L_\mathcal{S}^*(v) = P_\mathcal{S} L^*(v)$, where L^* is the adjoint of L and $P_\mathcal{S}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$. Show that the maximal subspace $\mathcal{W} \subseteq \mathcal{S}$ such that $L_\mathcal{S}^*|_\mathcal{W} = L^*|_\mathcal{W}$ satisfies*

$$\mathcal{W} = \left(\mathcal{S} \cap L\left(\mathcal{S}^\perp\right)\right)^{\perp_\mathcal{S}},$$

where $\mathcal{S}^\perp$ is the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}$ and the superscript $\perp_\mathcal{S}$ denotes the orthogonal complement in $\mathcal{S}$.

Exercise 39. *Prove that the metric on a Riemannian manifold M with a smooth foliation $\mathcal{F}$ is bundle-like if and only if the normal bundle $N\mathcal{F}$ with respect to that metric is totally geodesic.*

Exercise 40. *Let $(M, \mathcal{F})$ be a transversally oriented Riemannian foliation of codimension q with bundle-like metric, and let ν be the transversal volume form. The **transversal Hodge star operator** $\overline{\ast} : \wedge^* Q^* \to \wedge^* Q^*$ is defined by*

$$\alpha \wedge \overline{\ast}\beta = (\alpha, \beta)\, \nu,$$

for $\alpha, \beta \in \Omega^k(M, \mathcal{F})$, so that $\overline{\ast}1 = \nu$, $\overline{\ast}\nu = 1$. Let the transversal codifferential $\delta_T :$ be defined by

$$\delta_T = (-1)^{qk+q+1}\, \overline{\ast}d\overline{\ast} : \Omega^k(M, \mathcal{F}) \to \Omega^{k-1}(M, \mathcal{F}).$$

As above, let δ_b be the adjoint of d with respect to $L^2(\Omega(M, \mathcal{F}))$.
Prove the following identities:

- *$\overline{\ast}^2 = (-1)^{k(q-k)}$ on basic k-forms.*
- *If β is a basic one-form, then $(\beta\lrcorner) = (-1)^{q(k+1)}\,\overline{\ast}\,(\beta\wedge)\,\overline{\ast}$ as operators on basic k-forms.*
- *$\delta_b = \delta_T + \kappa_b\lrcorner$*
- *$\delta_b\nu = \overline{\ast}\kappa_b$*
- *$d\kappa_b = 0$ (Hint: compute $\delta_b^2\nu$.)*
- *$\overline{\ast}\widetilde{d} = \pm\widetilde{\delta}\overline{\ast}$, with $\widetilde{d} = d - \frac{1}{2}\kappa_b\wedge$, $\widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b\lrcorner$.*

3.2. The basic de Rham operator.
From the previous section, the basic de Rham operator is $D_b = \widetilde{d} + \widetilde{\delta}$ acting on basic forms, where

$$\widetilde{d} = d - \frac{1}{2}\kappa_b\wedge, \;\; \widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b\lrcorner.$$

Unlike the ordinary and well-studied basic Laplacian, the eigenvalues of $\widetilde{\Delta} = D_b^2$ are invariants of the Riemannian foliation structure alone and independent of the choice of compatible bundle-like metric. The operators $\widetilde{d}$ and $\widetilde{\delta}$ have following interesting properties.

Lemma 3.2. *$\widetilde{\delta}$ is the formal adjoint of $\widetilde{d}$.*

Lemma 3.3. *The maps $\widetilde{d}$ and $\widetilde{\delta}$ are differentials; that is, $\widetilde{d}^2 = 0$, $\widetilde{\delta}^2 = 0$. As a result, $\widetilde{d}$ and $\widetilde{\delta}$ commute with $\widetilde{\Delta} = D_b^2$, and $\ker\left(\widetilde{d} + \widetilde{\delta}\right) = \ker\left(\widetilde{\Delta}\right)$.*

Let $\Omega^k(M, \mathcal{F})$ denote the space of basic k-forms (either set of smooth forms or L^2-completion thereof), let $\widetilde{d}^k$ and $\widetilde{\delta_b}^k$ be the restrictions of $\widetilde{d}$ and $\widetilde{\delta_b}$ to k-forms, and let $\widetilde{\Delta}^k$ denote the restriction of D_b^2 to basic k-forms.

Proposition 3.4. *(Hodge decomposition) We have*

$$\Omega^k(M, \mathcal{F}) = \operatorname{image}\left(\widetilde{d}^{k-1}\right) \oplus \operatorname{image}\left(\widetilde{\delta_b}^{k+1}\right) \oplus \ker\left(\widetilde{\Delta}^k\right),$$

an L^2-orthogonal direct sum. Also, $\ker\left(\widetilde{\Delta}^k\right)$ is finite-dimensional and consists of smooth forms.

We call $\ker\left(\widetilde{\Delta}\right)$ the space $\widetilde{\Delta}$-harmonic forms. In the remainder of this section, we assume that the foliation is transversally oriented so that the transversal Hodge $\overline{*}$ operator is well-defined.

Definition 3.5. *We define the basic $\widetilde{d}$-cohomology $\widetilde{H}^*\left(M,\mathcal{F}\right)$ by*

$$\widetilde{H}^k\left(M,\mathcal{F}\right) = \frac{\ker\widetilde{d}^k}{\text{image }\widetilde{d}^{k-1}}.$$

The following proposition follows from standard arguments and the Hodge theorem (Theorem 3.4).

Proposition 3.6. *The finite-dimensional vector spaces $\widetilde{H}^k\left(M,\mathcal{F}\right)$ and $\ker\widetilde{\Delta}^k = \ker\left(\widetilde{d}+\widetilde{\delta}\right)^k$ are naturally isomorphic.*

We observe that for every choice of bundle-like metric, the differential $\widetilde{d} = d - \frac{1}{2}\kappa_b\wedge$ changes, and thus the cohomology groups change. However, note that κ_b is the only part that changes; for any two bundle-like metrics g_M, g'_M and associated κ_b, κ'_b compatible with $(M,\mathcal{F},g_Q)$, we have $\kappa'_b = \kappa_b + dh$ for some basic function h (see [2]). In the proof of the main theorem in [25], we essentially showed that the the basic de Rham operator D_b is then transformed by $D'_b = e^{h/2}D_b e^{-h/2}$. Applying this to our situation, we see that the $(\ker D'_b) = e^{h/2}\ker D_b$, and thus the cohomology groups are the same dimensions, independent of choices. To see this in our specific situation, note that if $\alpha \in \Omega^k\left(M,\mathcal{F}\right)$ satisfies $\widetilde{d}\alpha = 0$, then

$$
\begin{aligned}
\left(\widetilde{d}\right)'\left(e^{h/2}\alpha\right) &= \left(d - \frac{1}{2}\kappa_b\wedge - \frac{1}{2}dh\wedge\right)\left(e^{h/2}\alpha\right) \\
&= e^{h/2}d\alpha + \frac{1}{2}e^{h/2}dh\wedge\alpha - \frac{e^{h/2}}{2}\kappa_b\wedge\alpha - \frac{e^{h/2}}{2}dh\wedge\alpha \\
&= e^{h/2}d\alpha - \frac{e^{h/2}}{2}\kappa_b\wedge\alpha = e^{h/2}\left(d - \frac{1}{2}\kappa_b\wedge\right)\alpha = e^{h/2}\widetilde{d}\alpha = 0.
\end{aligned}
$$

Similarly, as in [25] one may show $\ker\left(\widetilde{\delta}\right)' = e^{h/2}\ker\left(\widetilde{\delta}\right)$, through a slightly more difficult computation. Thus, we have

Theorem 3.7. *(Conformal invariance of cohomology groups) Given a Riemannian foliation $(M,\mathcal{F},g_Q)$ and two bundle-like metrics g_M and g'_M compatible with g_Q, the $\widetilde{d}$-cohomology groups $\widetilde{H}^k\left(M,\mathcal{F}\right)$ are isomorphic, and that isomorphism is implemented by multiplication by a positive basic function. Further, the eigenvalues of the corresponding basic de Rham operators D_b and D'_b are identical, and the eigenspaces are isomorphic via multiplication by that same positive function.*

Corollary 3.8. *The dimensions of $\widetilde{H}^k\left(M,\mathcal{F}\right)$ and the eigenvalues of D_b (and thus of $\widetilde{\Delta} = D_b^2$) are invariants of the Riemannian foliation structure $(M,\mathcal{F},g_Q)$, independent of choice of compatible bundle-like metric g_M.*

Exercise 41. *Show that if α is any closed form, then $(d + \alpha\wedge)^2 = 0$.*

Exercise 42. *Show that $\widetilde{\delta}$ is the formal adjoint of $\widetilde{d}$.*

Exercise 43. *Show that a Riemannian foliation $(M, \mathcal{F})$ is taut if and only if $\widetilde{H}^0(M, \mathcal{F})$ is nonzero. (A Riemannian foliation is **taut** if there exists a bundle-like metric for which the leaves are minimal submanifolds and thus have zero mean curvature.)*

Exercise 44. *(This example is contained in* [26]*.) This Riemannian foliation is the famous Carrière example from* [14] *in the 3-dimensional case. Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ of trace strictly greater than 2. We denote respectively by V_1 and V_2 the eigenvectors associated with the eigenvalues λ and $\frac{1}{\lambda}$ of A with $\lambda > 1$ irrational. Let the hyperbolic torus $\mathbb{T}_A^3$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(A(m), t+1)$. The flow generated by the vector field V_2 is a transversally Lie foliation of the affine group. We denote by K the holonomy subgroup. The affine group is the Lie group $\mathbb{R}^2$ with multiplication $(t, s).(t', s') = (t + t', \lambda^t s' + s)$, and the subgroup K is*

$$K = \{(n, s), n \in \mathbb{Z}, s \in \mathbb{R}\}.$$

We choose the bundle-like metric (letting (x, s, t) denote the local coordinates in the V_2 direction, V_1 direction, and $\mathbb{R}$ direction, respectively) as

$$g = \lambda^{-2t} dx^2 + \lambda^{2t} ds^2 + dt^2.$$

Prove that:
- *The mean curvature of the flow is $\kappa = \kappa_b = \log(\lambda)\, dt$.*
- *The twisted basic cohomology groups are all trivial.*
- *The ordinary basic cohomology groups satisfy $H^0(M, \mathcal{F}) \cong \mathbb{R}$, $H^1(M, \mathcal{F}) \cong \mathbb{R}$, $H^2(M, \mathcal{F}) \cong \{0\}$.*
- *The flow is not taut.*

3.3. Poincaré duality and consequences.

Theorem 3.9. *(Poincaré duality for $\widetilde{d}$-cohomology) Suppose that the Riemannian foliation $(M, \mathcal{F}, g_Q)$ is transversally oriented and is endowed with a bundle-like metric. For each k such that $0 \le k \le q$ and any compatible choice of bundle-like metric, the map $\bar{\ast} : \Omega^k(M, \mathcal{F}) \to \Omega^{q-k}(M, \mathcal{F})$ induces an isomorphism on the $\widetilde{d}$-cohomology. Moreover, $\bar{\ast}$ maps the $\ker \widetilde{\Delta}^k$ isomorphically onto $\ker \widetilde{\Delta}^{q-k}$, and it maps the λ-eigenspace of $\widetilde{\Delta}^k$ isomorphically onto the λ-eigenspace of $\widetilde{\Delta}^{q-k}$, for all $\lambda \ge 0$.*

This resolves the problem of the failure of Poincaré duality to hold for standard basic cohomology (see [34], [56]).

Corollary 3.10. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to the $\widetilde{H}^*(M, \mathcal{F})$ vanishes.*

The following fact is a new result for ordinary basic cohomology of Riemannian foliations. Ordinary basic cohomology does not satisfy Poincaré duality; in fact, the top-dimensional basic cohomology group is zero if and only if the foliation is not taut. Also, leaf closures of a transversally oriented foliation can fail to be transversally oriented, so orientation is also a tricky issue.

Corollary 3.11. *Let $(M, \mathcal{F})$ be a smooth transversally oriented foliation of odd codimension that admits a transverse Riemannian structure. Then the Euler characteristic associated to the ordinary basic cohomology $H^*(M, \mathcal{F})$ vanishes.*

Proof. The basic Euler characteristic is the basic index of the operator $D_0 = d + \delta_B :$ $\Omega^{\mathrm{even}}(M, \mathcal{F}) \to \Omega^{\mathrm{odd}}(M, \mathcal{F})$. See [12], [18], [7], [19] for information on the basic index and basic Euler characteristic. The crucial property for us is that the basic index of D_0 is a Fredholm index and is invariant under perturbations of the operator through transversally elliptic operators that map the basic forms to themselves. In particular, the family of operators $D_t = d + \delta_b - \frac{t}{2}\kappa_b \lrcorner - \frac{t}{2}\kappa_b \wedge$ for $0 \leq t \leq 1$ meets that criteria, and $D_1 = D_b$ is the basic de Rham operator $D_b : \Omega^{\mathrm{even}}(M, \mathcal{F}) \to \Omega^{\mathrm{odd}}(M, \mathcal{F})$. Thus, the basic Euler characteristic of the basic cohomology complex is the same as the basic Euler characteristic of the $\widetilde{d}$-cohomology complex. The result follows from the previous corollary. $\qquad\square$

Exercise 45. *Prove that the twisted cohomology class $[\kappa_b]$ is always trivial.*

Exercise 46. *Prove that if $(M, \mathcal{F})$ is not taut, then the ordinary basic cohomology satisfies $\dim H^1(M, \mathcal{F}) \geq 1$.*

Exercise 47. *Prove that there exists a monomorphism from $H^1(M, \mathcal{F})$ to $H^1(M)$.*

Exercise 48. *True or false: $\dim H^2(M, \mathcal{F}) \leq \dim H^2(M)$.*

Exercise 49. *Under what conditions is it true that $\dim \widetilde{H}^1(M, \mathcal{F}) \geq \dim H^1(M, \mathcal{F})$?*

Exercise 50. *(Hard) Find an example of a Riemannian foliation that is not taut and whose twisted basic cohomology is nontrivial.*
(If you give up, find an answer in [26].)

4. Natural examples of transversal Dirac operators on G-manifolds

The research content of this section is joint work with I. Prokhorenkov, from [45].

4.1. **Equivariant structure of the orthonormal frame bundle.** We first make the important observation that if a Lie group acts effectively by isometries on a Riemannian manifold, then this action can be lifted to a free action on the orthonormal frame bundle. Given a complete, connected G-manifold, the action of $g \in G$ on M induces an action of dg on TM, which in turn induces an action of G on the principal $O(n)$-bundle $F_O \xrightarrow{p} M$ of orthonormal frames over M.

Lemma 4.1. *The action of G on F_O is regular, and the isotropy subgroups corresponding to any two points of F_O are the same.*

Proof. Let H be the isotropy subgroup of a frame $f \in F_O$. Then H also fixes $p(f) \in M$, and since H fixes the frame, its differentials fix the entire tangent space at $p(f)$. Since it fixes the tangent space, every element of H also fixes every frame in $p^{-1}(p(f))$; thus every frame in a given fiber must have the same isotropy subgroup. Since the elements of H map geodesics to geodesics and preserve distance, a neighborhood of $p(f)$ is fixed by H. Thus, H is a subgroup of the isotropy subgroup at each point of that neighborhood. Conversely, if an element of G fixes a neighborhood of a point x in M, then it fixes all frames in $p^{-1}(x)$, and thus all frames in the fibers above that neighborhood. Since M is connected, we may conclude that every point of F_O has the same isotropy subgroup H, and H is the subgroup of G that fixes every point of M. $\qquad\square$

Remark 4.2. *Since this subgroup H is normal, we often reduce the group G to the group G/H so that our action is effective, in which case the isotropy subgroups on F_O are all trivial.*

Remark 4.3. *A similar idea is also useful in constructing the lifted foliation and the basic manifold associated to a Riemannian foliation (see* [43]*).*

In any case, the G orbits on F_O are diffeomorphic and form a Riemannian fiber bundle, in the natural metric on F_O defined as follows. The Levi-Civita connection on M determines the horizontal subbundle $\mathcal{H}$ of TF_O. We construct the local product metric on F_O using a biinvariant fiber metric and the pullback of the metric on M to $\mathcal{H}$; with this metric, F_O is a compact Riemannian $G \times O(n)$-manifold. The lifted G-action commutes with the $O(n)$-action. Let $\mathcal{F}$ denote the foliation of G-orbits on F_O, and observe that $F_O \xrightarrow{\pi} F_O/G = F_O/\mathcal{F}$ is a Riemannian submersion of compact $O(n)$-manifolds.

Let $E \to F_O$ be a Hermitian vector bundle that is equivariant with respect to the $G \times O(n)$ action. Let $\rho : G \to U(V_\rho)$ and $\sigma : O(n) \to U(W_\sigma)$ be irreducible unitary representations. We define the bundle $\mathcal{E}^\sigma \to M$ by

$$\mathcal{E}^\sigma_x = \Gamma\left(p^{-1}(x), E\right)^\sigma,$$

where the superscript σ is defined for a $O(n)$-module Z by

$$Z^\sigma = \mathrm{eval}\left(\mathrm{Hom}_{O(n)}(W_\sigma, Z) \otimes W_\sigma\right),$$

where eval : $\mathrm{Hom}_{O(n)}(W_\sigma, Z) \otimes W_\sigma \to Z$ is the evaluation map $\phi \otimes w \mapsto \phi(w)$. The space Z^σ is the vector subspace of Z on which $O(n)$ acts as a direct sum of representations of type σ. The bundle $\mathcal{E}^\sigma$ is a Hermitian G-vector bundle of finite rank over M. The metric on $\mathcal{E}^\sigma$ is chosen as follows. For any $v_x, w_x \in \mathcal{E}^\sigma_x$, we define

$$\langle v_x, w_x \rangle := \int_{p^{-1}(x)} \langle v_x(y), w_x(y) \rangle_{y,E} \; d\mu_x(y),$$

where $d\mu_x$ is the measure on $p^{-1}(x)$ induced from the metric on F_O. See [12] for a similar construction.

Similarly, we define the bundle $\mathcal{T}^\rho \to F_O/G$ by

$$\mathcal{T}^\rho_y = \Gamma\left(\pi^{-1}(y), E\right)^\rho,$$

and $\mathcal{T}^\rho \to F_O/G$ is a Hermitian $O(n)$-equivariant bundle of finite rank. The metric on $\mathcal{T}^\rho$ is

$$\langle v_z, w_z \rangle := \int_{\pi^{-1}(y)} \langle v_z(y), w_z(y) \rangle_{z,E} \; dm_z(y),$$

where dm_z is the measure on $\pi^{-1}(z)$ induced from the metric on F_O.

The vector spaces of sections $\Gamma(M, \mathcal{E}^\sigma)$ and $\Gamma(F_O, E)^\sigma$ can be identified via the isomorphism

$$i_\sigma : \Gamma(M, \mathcal{E}^\sigma) \to \Gamma(F_O, E)^\sigma,$$

where for any section $s \in \Gamma(M, \mathcal{E}^\sigma)$, $s(x) \in \Gamma(p^{-1}(x), E)^\sigma$ for each $x \in M$, and we let

$$i_\sigma(s)(f_x) := s(x)|_{f_x}$$

for every $f_x \in p^{-1}(x) \subset F_O$. Then $i_\sigma^{-1} : \Gamma(F_O, E)^\sigma \to \Gamma(M, \mathcal{E}^\sigma)$ is given by

$$i_\sigma^{-1}(u)(x) = u|_{p^{-1}(x)}.$$

Observe that $i_\sigma : \Gamma(M, \mathcal{E}^\sigma) \to \Gamma(F_O, E)^\sigma$ extends to an L^2 isometry. Given $u, v \in \Gamma(M, \mathcal{E}^\sigma)$,

$$
\begin{aligned}
\langle u, v \rangle_M &= \int_M \langle u_x, v_x \rangle \ dx = \int_M \int_{p^{-1}(x)} \langle u_x(y), v_x(y) \rangle_{y,E} \ d\mu_x(y) \ dx \\
&= \int_M \left(\int_{p^{-1}(x)} \langle i_\sigma(u), i_\sigma(v) \rangle_E \ d\mu_x(y) \right) dx \\
&= \int_{F_O} \langle i_\sigma(u), i_\sigma(v) \rangle_E = \langle i_\sigma(u), i_\sigma(v) \rangle_{F_O},
\end{aligned}
$$

where dx is the Riemannian measure on M; we have used the fact that p is a Riemannian submersion. Similarly, we let

$$
j_\rho : \Gamma(F_O/G, \mathcal{T}^\rho) \to \Gamma(F_O, E)^\rho
$$

be the natural identification, which extends to an L^2 isometry.

Let

$$
\Gamma(M, \mathcal{E}^\sigma)^\alpha = \mathrm{eval}\left(\mathrm{Hom}_G(V_\alpha, \Gamma(M, \mathcal{E}^\sigma)) \otimes V_\alpha\right).
$$

Similarly, let

$$
\Gamma(F_O/G, \mathcal{T}^\rho)^\beta = \mathrm{eval}\left(\mathrm{Hom}_G(W_\beta, \Gamma(F_O/G, \mathcal{T}^\rho)) \otimes W_\beta\right).
$$

Theorem 4.4. *For any irreducible representations $\rho : G \to U(V_\rho)$ and $\sigma : O(n) \to U(W_\sigma)$, the map $j_\rho^{-1} \circ i_\sigma : \Gamma(M, \mathcal{E}^\sigma)^\rho \to \Gamma(F_O/G, \mathcal{T}^\rho)^\sigma$ is an isomorphism (with inverse $i_\sigma^{-1} \circ j_\rho$) that extends to an L^2-isometry.*

Exercise 51. *Prove that if M is a Riemannian manifold, then the orthonormal frame bundle of M has trivial tangent bundle.*

Exercise 52. *Suppose that a compact Lie group acts smoothly on a Riemannian manifold. Prove that there exists a metric on the manifold such that the Lie group acts isometrically.*

Exercise 53. *Let $\mathbb{Z}_2$ act on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with an action generated by $(x, y) = (-x, y)$ for $x, y \in \mathbb{R}/\mathbb{Z}$.*

- *Find the quotient space $T^2/\mathbb{Z}_2$.*
- *Find all the irreducible representations of $\mathbb{Z}_2$. (Hint: they are all homomorphisms $\rho : \mathbb{Z}_2 \to U(1)$.)*
- *Find the orthonormal frame bundle F_O, and determine the induced action of $\mathbb{Z}_2$ on F_O.*
- *Find the quotient space $F_O/\mathbb{Z}_2$, and determine the induced action of $O(2)$ on this manifold.*

Exercise 54. *Suppose that $M = S^2$ is the unit sphere in $\mathbb{R}^3$. Let S^1 act on S^2 by rotations around the x_3-axis.*

- *Show that the oriented orthonormal frame bundle F_{SO} can be identified with $SO(3)$, which in turn can be identified with $\mathbb{R}P^3$.*
- *Show that the lifted S^1 action on F_{SO} can be realized by the orbits of a left-invariant vector field on $SO(3)$.*
- *Find the quotient F_{SO}/S^1.*

Exercise 55. *Suppose that a compact, connected Lie group acts by isometries on a Riemannian manifold. Show that all harmonic forms are invariant under pullbacks by the group action.*

4.2. **Dirac-type operators on the frame bundle.** Let $E \to F_O$ be a Hermitian vector bundle of $\mathbb{Cl}(N\mathcal{F})$ modules that is equivariant with respect to the $G \times O(n)$ action. With notation as in previous sections, we have the transversal Dirac operator $A_{N\mathcal{F}}$ defined by the composition

$$\Gamma(F_O, E) \xrightarrow{\nabla} \Gamma(F_O, T^*F_O \otimes E) \xrightarrow{\text{proj}} \Gamma(F_O, N^*\mathcal{F} \otimes E) \xrightarrow{c} \Gamma(F_O, E).$$

As explained previously, the operator

$$D_{N\mathcal{F}} = A_{N\mathcal{F}} - \frac{1}{2}c(H)$$

is a essentially self-adjoint $G \times O(n)$-equivariant operator, where H is the mean curvature vector field of the G-orbits in F_O.

From $D_{N\mathcal{F}}$ we now construct equivariant differential operators on M and F_O/G, as follows. We define the operators

$$D_M^\sigma := i_\sigma^{-1} \circ D_{N\mathcal{F}} \circ i_\sigma : \Gamma(M, \mathcal{E}^\sigma) \to \Gamma(M, \mathcal{E}^\sigma),$$

and

$$D_{F_O/G}^\rho := j_\rho^{-1} \circ D_{N\mathcal{F}} \circ j_\rho : \Gamma(F_O/G, \mathcal{T}^\rho) \to \Gamma(F_O/G, \mathcal{T}^\rho).$$

For an irreducible representation $\alpha : G \to U(V_\alpha)$, let

$$(D_M^\sigma)^\alpha : \Gamma(M, \mathcal{E}^\sigma)^\alpha \to \Gamma(M, \mathcal{E}^\sigma)^\alpha$$

be the restriction of D_M^σ to sections of G-representation type $[\alpha]$. Similarly, for an irreducible representation $\beta : G \to U(W_\beta)$, let

$$\left(D_{F_O/G}^\rho\right)^\beta : \Gamma(F_O/G, \mathcal{T}^\rho)^\beta \to \Gamma(F_O/G, \mathcal{T}^\rho)^\beta$$

be the restriction of $D_{F_O/G}^\rho$ to sections of $O(n)$-representation type $[\beta]$. The proposition below follows from Theorem 4.4.

Proposition 4.5. *The operator D_M^σ is transversally elliptic and G-equivariant, and $D_{F_O/G}^\rho$ is elliptic and $O(n)$-equivariant, and the closures of these operators are self-adjoint. The operators $(D_M^\sigma)^\rho$ and $\left(D_{F_O/G}^\rho\right)^\sigma$ have identical discrete spectrum, and the corresponding eigenspaces are conjugate via Hilbert space isomorphisms.*

Thus, questions about the transversally elliptic operator D_M^σ can be reduced to questions about the elliptic operators $D_{F_O/G}^\rho$ for each irreducible $\rho : G \to U(V_\rho)$.

In particular, we are interested in the equivariant index, which we will explain in detail the next section. In the following theorem, $\text{ind}^G(\cdot)$ denotes the virtual representation-valued index as explained in [3] and in Section 5.1; the result is a formal difference of finite-dimensional representations if the input is a symbol of an elliptic operator.

Theorem 4.6. *Suppose that F_O is G-transversally spin^c. Then for every transversally elliptic symbol class $[u] \in K_{cpt,G}(T_G^*M)$, there exists an operator of type D_M^1 such that $\text{ind}^G(u) = \text{ind}^G(D_M^1)$.*

Exercise 56. *(continuation of Exercise 53)*

- *Determine a Dirac operator on the trivial $\mathbb{C}^2$ bundle over the three-dimensional F_O. (Hint: use the Dirac operator from $\mathbb{R}^3$.)*

- *Find the induced operator $D^{\mathbf{1}}_{T^2}$ on T^2, where $\mathbf{1}$ denotes the trivial representation $\mathbf{1} : O(2) \to \{1\} \in U(1)$. This means the restriction of the Dirac operator of F_O to sections that are invariant under the $O(2)$ action.*
- *Identify all irreducible unitary representations of $O(2)$. (Hint: they are all homomorphisms $\sigma : O(2) \to U(1)$.)*
- *Find $\ker D^{\mathbf{1}}_{T^2}$ and $\ker D^{\mathbf{1}*}_{T^2}$, and decompose these vector spaces as direct sums of irreducible unitary representations of $O(2)$.*
- *For each irreducible unitary representation $\rho : \mathbb{Z}_2 \to U(1)$ of $\mathbb{Z}_2$, determine the induced operator $D^{\rho}_{F_O/\mathbb{Z}_2}$.*

Exercise 57. *(continuation of Exercise 54)*

- *Starting with a transversal Dirac operator on the trivial $\mathbb{C}^2$ bundle over $SO(3)$, find the induced operator $D^{\mathbf{1}}_{S^2}$ on S^2.*
- *Identify all irreducible unitary representations of S^1.*
- *Find $\ker D^{\mathbf{1}}_{S^2}$ and $\ker D^{\mathbf{1}*}_{S^2}$, and decompose these vector spaces as direct sums of irreducible unitary representations of S^1.*

Exercise 58. *(continuation of Exercise 55) Suppose that a compact, connected Lie group G acts on a Riemannian manifold. Show that the $\ker(d + \delta)$ is the same as G-invariant part $\ker(d + \delta)^{\mathbf{1}}$ of $\ker(d + \delta)$, and $\ker(d + \delta)^{\rho} = 0$ for all other irreducible representations $\rho : G \to U(V_\rho)$.*

Exercise 59. *Suppose that $G = M$ acts freely on itself. Construct a transversal Dirac operator acting on a trivial spinor bundle on the orthonormal frame bundle. Determine the operator $D^{\mathbf{1}}_G$ for $\mathbf{1}$ being the trivial representation of $O(n)$, and find $\ker D^{\mathbf{1}}_G$ and $\ker D^{\mathbf{1}*}_G$.*

5. Transverse index theory for G-manifolds and Riemannian foliations

The research content and some of the expository content in this section are joint work with J. Brüning and F. W. Kamber, from [12] and [13].

5.1. **Introduction: the equivariant index.** Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M, and let $E = E^+ \oplus E^-$ be a graded, G-equivariant Hermitian vector bundle over M. We consider a first order G-equivariant differential operator $D = D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ which is transversally elliptic (as explained at the beginning of Section 53). Let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M, E^{\pm})$ by $(gs)(x) = g \cdot s(g^{-1}x)$, and the (possibly infinite-dimensional) subspaces $\ker(D)$ and $\ker(D^*)$ are G-invariant subspaces. Let $\rho : G \to U(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho = \mathrm{tr}(\rho)$ denote its character. Let $\Gamma(M, E^{\pm})^{\rho}$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^{\pm})$ that are unitarily equivalent to the ρ representation. The operators
$$D^{\rho} : \Gamma(M, E^+)^{\rho} \to \Gamma(M, E^-)^{\rho}$$
can be extended to be Fredholm operators, so that each irreducible representation of G appears with finite multiplicity in $\ker D^{\pm}$. Let $a^{\pm}_{\rho} \in \mathbb{Z}^+$ be the multiplicity of ρ in $\ker(D^{\pm})$.

The virtual representation-valued index of D (see [3]) is
$$\mathrm{ind}^G(D) := \sum_{\rho} (a^+_{\rho} - a^-_{\rho})[\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\text{ind}^\rho(D) := a_\rho^+ - a_\rho^- = \frac{1}{\dim V_\rho}\text{ind}\left(D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}\right).$$

In particular, if $\mathbf{1}$ is the trivial representation of G, then

$$\text{ind}^{\mathbf{1}}(D) = \text{ind}\left(D|_{\Gamma(M,E^+)^G \to \Gamma(M,E^-)^G}\right),$$

where the superscript G implies restriction to G-invariant sections.

There is a relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\text{ind}_g(D)$ (see [3]); the multiplicities determine the distributional index, and vice versa. Because the operator $D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}$ is Fredholm, all of the indices $\text{ind}^G(D)$, $\text{ind}_g(D)$, and $\text{ind}^\rho(D)$ depend only on the homotopy class of the principal transverse symbol of D.

The new equivariant index result (in [12])is stated in Theorem 5.13. A large body of work over the last twenty years has yielded theorems that express $\text{ind}_g(D)$ in terms of topological and geometric quantities (as in the Atiyah-Segal index theorem for elliptic operators or the Berline-Vergne Theorem for transversally elliptic operators — see [4],[8],[9]). However, until now there has been very little known about the problem of expressing $\text{ind}^\rho(D)$ in terms of topological or geometric quantities which are determined at the different strata

$$M([H]) := \bigcup_{G_x \in [H]} x$$

of the G-manifold M. The special case when all of the isotropy groups are finite was solved by M. Atiyah in [3], and this result was utilized by T. Kawasaki to prove the Orbifold Index Theorem (see [37]). Our analysis is new in that the equivariant heat kernel related to the index is integrated first over the group and second over the manifold, and thus the invariants in our index theorem (Theorem 5.13) are very different from those seen in other equivariant index formulas. The explicit nature of the formula is demonstrated in Theorem 5.14, a special case where the equivariant Euler characteristic is computed in terms of invariants of the G-manifold strata.

One of the primary motivations for obtaining an explicit formula for $\text{ind}^\rho(D)$ was to use it to produce a basic index theorem for Riemannian foliations, thereby solving a problem that has been open since the 1980s (it is mentioned, for example, in [19]). In fact, the basic index theorem (Theorem 5.16) is a consequence of the equivariant index theorem. We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations. Using a different technique, they are able to prove a formula for the basic index of a basic Dirac operator that is distinct from our formula, in the case where all the infinitessimal holonomy groups of the foliation are connected tori and if Molino's commuting sheaf is abelian and has trivial holonomy (see [23]). Our result requires at most mild topological assumptions on the transverse structure of the strata of the Riemannian foliation and has a similar form to the formula above for $\text{ind}^{\mathbf{1}}(D)$. In particular, the analogue for the Gauss-Bonnet Theorem for Riemannian foliations (Theorem 5.17) is a corollary and requires no assumptions on the structure of the Riemannian foliation.

There are several new techniques used in the proof of the equivariant index theorem that have not been explored previously, and we will briefly describe them in upcoming sections. First, the proof requires a modification of the equivariant structure. In Section 5.2, we

describe the known structure of G-manifolds. In Section 5.3, we describe a process of blowing up, cutting, and reassembling the G-manifold into what is called the *desingularization*. The result is a G-manifold that has less intricate structure and for which the analysis is more simple. We note that our desingularization process and the equivariant index theorem were stated and announced in [50] and [51]; recently Albin and Melrose have taken it a step further in tracking the effects of the desingularization on equivariant cohomology and equivariant K-theory ([1]).

Another crucial step in the proof of the equivariant index theorem is the decomposition of equivariant vector bundles over G-manifolds with one orbit type. We construct a subbundle of an equivariant bundle over a G-invariant part of a stratum that is the minimal G-bundle decomposition that consists of direct sums of isotypical components of the bundle. We call this decomposition the *fine decomposition* and define it in Section 5.4. More detailed accounts of this method are in [12], [30].

Exercise 60. *Let* $Z : C^\infty(S^1, \mathbb{C}) \to \{0\}$ *denote the zero operator on complex-valued functions on the circle* S^1. *If we consider* Z *to be an* S^1-*equivariant operator on the circle, find* $\mathrm{ind}^\rho(Z)$ *for every irreducible representation* $\rho : S^1 \to U(1)$. *(Important: the target bundle is the zero vector bundle). We are assuming that* S^1 *acts by rotations.*

Exercise 61. *In Exercise 60, instead calculate each* $\mathrm{ind}^\rho(Z)$, *if* $Z : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$ *is multiplication by zero.*

Exercise 62. *Let* $D = i\frac{d}{d\theta} : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$ *be an operator on complex-valued functions on the circle* $S^1 = \{e^{i\theta} : \theta \in \mathbb{R} / 2\pi\mathbb{Z}\}$. *Consider* D *to be a* $\mathbb{Z}_2$-*equivariant operator, where the action is generated by* $\theta \mapsto \theta + \pi$. *Find* $\mathrm{ind}^\rho(Z)$ *for every irreducible representation* $\rho : \mathbb{Z}_2 \to U(1)$.

Exercise 63. *Let* $D = i\frac{d}{d\theta} : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$ *be an operator on complex-valued functions on the circle* $S^1 = \{e^{i\theta} : \theta \in \mathbb{R} / 2\pi\mathbb{Z}\}$. *Consider the* $\mathbb{Z}_2$ *action generated by* $\theta \mapsto -\theta$. *Show that* D *is not* $\mathbb{Z}_2$-*equivariant.*

Exercise 64. *Let* $\mathbb{Z}_2$ *act on* $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ *with an action generated by* $(x, y) = (-x, y)$ *for* $x, y \in \mathbb{R} / \mathbb{Z}$. *Calculate* $\mathrm{ind}^\rho(D)$ *for every irreducible representation* $\rho : \mathbb{Z}_2 \to U(1)$, *where* D *is the standard Dirac operator on the trivial* $\mathbb{C}^2$ *bundle.*

5.2. **Stratifications of G-manifolds.** In the following, we will describe some standard results from the theory of Lie group actions (see [11], [36]). As above, G is a compact Lie group acting on a smooth, connected, closed manifold M. We assume that the action is effective, meaning that no $g \in G$ fixes all of M. (Otherwise, replace G with $G / \{g \in G : gx = x \text{ for all } x \in M\}$.) Choose a Riemannian metric for which G acts by isometries.

Given such an action and $x \in M$, the **isotropy or stabilizer subgroup** $G_x < G$ is defined to be $\{g \in G : gx = x\}$. The **orbit** $\mathcal{O}_x$ of a point x is defined to be $\{gx : g \in G\}$. Since $G_{xg} = gG_xg^{-1}$, the conjugacy class of the isotropy subgroup of a point is fixed along an orbit.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the **orbit type**. On any such G-manifold, there are a finite number of orbit types, and there is a partial order on the set of orbit types. Given subgroups H and K of G, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K, and we say $[H] < [K]$ if $[H] \leq [K]$ and

$[H] \neq [K]$. We may enumerate the conjugacy classes of isotropy subgroups as $[G_0], ..., [G_r]$ such that $[G_i] \leq [G_j]$ implies that $i \leq j$. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other orbits are called **singular**. As a consequence, every isotropy subgroup H satisfies $[G_0] \leq [H]$. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the **stratum** corresponding to $[G_j]$. If $[G_j] \leq [G_k]$, it follows that the closure of M_j contains the closure of M_k. A stratum M_j is called a **minimal stratum** if there does not exist a stratum M_k such that $[G_j] < [G_k]$ (equivalently, such that $\overline{M_k} \subsetneq \overline{M_j}$). It is known that each stratum is a G-invariant submanifold of M, and in fact a minimal stratum is a closed (but not necessarily connected) submanifold. Also, for each j, the submanifold $M_{\geq j} := \bigcup_{[G_k] \geq [G_j]} M_k$ is a closed, G-invariant submanifold.

Now, given a proper, G-invariant submanifold S of M and $\varepsilon > 0$, let $T_\varepsilon(S)$ denote the union of the images of the exponential map at s for $s \in S$ restricted to the open ball of radius ε in the normal bundle at S. It follows that $T_\varepsilon(S)$ is also G-invariant. If M_j is a stratum and ε is sufficiently small, then all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. This implies that if $j < k$, $\overline{M_j} \cap \overline{M_k} \neq \varnothing$, and $M_k \subsetneq \overline{M_j}$, then $\overline{M_j}$ and $\overline{M_k}$ intersect at right angles, and their intersection consists of more singular strata (with isotropy groups containing conjugates of both G_k and G_j).

Fix $\varepsilon > 0$. We now decompose M as a disjoint union of sets $M_0^\varepsilon, \ldots, M_r^\varepsilon$. If there is only one isotropy type on M, then $r = 0$, and we let $M_0^\varepsilon = \Sigma_0^\varepsilon = M_0 = M$. Otherwise, for $j = r, r-1, ..., 0$, let $\varepsilon_j = 2^j \varepsilon$, and let

$$\Sigma_j^\varepsilon = M_j \setminus \overline{\bigcup_{k>j} M_k^\varepsilon}; \ M_j^\varepsilon = T_{\varepsilon_j}(M_j) \setminus \overline{\bigcup_{k>j} M_k^\varepsilon}.$$

Thus,

$$T_\varepsilon\left(\Sigma_j^\varepsilon\right) \subset M_j^\varepsilon, \ \Sigma_j^\varepsilon \subset M_j.$$

The following facts about this decomposition are contained in [36, pp. 203ff]:

Lemma 5.1. *For sufficiently small $\varepsilon > 0$, we have, for every $i \in \{0, \ldots, r\}$:*

(1) *$M = \coprod_{i=0}^r M_i^\varepsilon$ (disjoint union).*

(2) *M_i^ε is a union of G-orbits; Σ_i^ε is a union of G-orbits.*

(3) *The manifold M_i^ε is diffeomorphic to the interior of a compact G-manifold with corners; the orbit space M_i^ε/G is a smooth manifold that is isometric to the interior of a triangulable, compact manifold with corners. The same is true for each Σ_i^ε.*

(4) *If $[G_j]$ is the isotropy type of an orbit in M_i^ε, then $j \leq i$ and $[G_j] \leq [G_i]$.*

(5) *The distance between the submanifold M_j and M_i^ε for $j > i$ is at least ε.*

Exercise 65. *Suppose G and M are as above. Show that if γ is a geodesic that is perpendicular at $x \in M$ to the orbit $\mathcal{O}_x$ through x, then γ is perpendicular to every orbit that intersects it.*

Exercise 66. *With G and M as above, suppose that γ is a geodesic that is orthogonal to a particular singular stratum Σ. Prove that each element $g \in G$ maps γ to another geodesic with the same property.*

Exercise 67. *Prove that if S is any set of isometries of a Riemannian manifold M, then the fixed point set $M^S := \{x \in M : gx = x$ for every $g \in S\}$ is a totally geodesic submanifold of M.*

Exercise 68. *Prove that if Σ is a stratum of the action of G on M corresponding to isotropy type $[H]$, then the fixed point set Σ^H is a principal $N(H) / H$ bundle over $G \backslash \Sigma$, where $N(H)$ is the normalizer of the subgroup H.*

Exercise 69. *Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ act on $S^2 \subset \mathbb{R}^3$ via $(x, y, z) \mapsto (-x, y, z)$ and $(x, y, z) \mapsto (x, -y, z)$. Determine the strata of this action and the isotropy types.*

Exercise 70. *Let $O(2)$ act on $S^2 \subset \mathbb{R}^3$ by rotations that fix the z-axis. Determine the strata of this action and the isotropy types.*

Exercise 71. *Let M be the rectangle $[0, 1] \times [-1, 1]$ along with identifications $(s, 1) \sim (s, -1)$ for $0 \le s \le 1$, $(0, x) \sim \left(0, x + \frac{1}{2}\right)$ for $0 \le x \le \frac{1}{2}$, and $(1, x) \sim \left(1, x + \frac{1}{2}\right)$ for $0 \le x \le \frac{1}{2}$.*
- *Show that M is a smooth Riemannian manifold when endowed with the standard flat metric.*
- *Find the topological type of the surface M.*
- *Suppose that $S^1 = \mathbb{R} / 2\mathbb{Z}$ acts on M via $(s, x) \mapsto (s, x + t)$, with $x, t \in \mathbb{R} / 2\mathbb{Z}$. Find the strata and the isotropy types of this action.*

5.3. Equivariant desingularization.
Assume that G is a compact Lie group that acts on a Riemannian manifold M by isometries. We construct a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the **desingularization** of M and is denoted $\widetilde{M}$. We also refer to [1] for their recent related explanation of this desingularization (which they call *resolution*).

A sequence of modifications is used to construct N and $\widetilde{M} \subset N$. Let M_j be a minimal stratum. Let $T_\varepsilon(M_j)$ denote a tubular neighborhood of radius ε around M_j, with ε chosen sufficiently small so that all orbits in $T_\varepsilon(M_j) \setminus M_j$ are of type $[G_k]$, where $[G_k] < [G_j]$. Let

$$N^1 = (M \setminus T_\varepsilon(M_j)) \cup_{\partial T_\varepsilon(M_j)} (M \setminus T_\varepsilon(M_j))$$

be the manifold constructed by gluing two copies of $(M \setminus T_\varepsilon(M_j))$ smoothly along the boundary (the codimension one case should be treated in a slightly different way; see [12] for details). Since the $T_\varepsilon(M_j)$ is saturated (a union of G-orbits), the G-action lifts to N^1. Note that the strata of the G-action on N^1 correspond to strata in $M \setminus T_\varepsilon(M_j)$. If $M_k \cap (M \setminus T_\varepsilon(M_j))$ is nontrivial, then the stratum corresponding to isotropy type $[G_k]$ on N^1 is

$$N_k^1 = (M_k \cap (M \setminus T_\varepsilon(M_j))) \cup_{(M_k \cap \partial T_\varepsilon(M_j))} (M_k \cap (M \setminus T_\varepsilon(M_j))).$$

Thus, N^1 is a G-manifold with one fewer stratum than M, and $M \setminus M_j$ is diffeomorphic to one copy of $(M \setminus T_\varepsilon(M_j))$, denoted $\widetilde{M}^1$ in N^1. In fact, N^1 is a branched double cover of M, branched over M_j. If N^1 has one orbit type, then we set $N = N^1$ and $\widetilde{M} = \widetilde{M}^1$. If N^1 has more than one orbit type, we repeat the process with the G-manifold N^1 to produce a new G-manifold N^2 with two fewer orbit types than M and that is a 4-fold branched cover of M. Again, $\widetilde{M}^2$ is a fundamental domain of $\widetilde{M}^1 \setminus \{a$ minimal stratum$\}$, which is a fundamental domain of M with two strata removed. We continue until $N = N^r$ is a G-manifold with all orbits of type $[G_0]$ and is a 2^r-fold branched cover of M, branched over $M \setminus M_0$. We set $\widetilde{M} = \widetilde{M}^r$, which is a fundamental domain of M_0 in N.

Further, one may independently desingularize $M_{\geq j}$, since this submanifold is itself a closed G-manifold. If $M_{\geq j}$ has more than one connected component, we may desingularize all components simultaneously. The isotropy type of all points of $\widetilde{M_{\geq j}}$ is $[G_j]$, and $\widetilde{M_{\geq j}}/G$ is a smooth (open) manifold.

Exercise 72. *Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 69.*

Exercise 73. *Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 70.*

Exercise 74. *Find the desingularization $\widetilde{M_j}$ of each stratum M_j for the G-manifold in Exercise 71.*

5.4. **The fine decomposition of an equivariant bundle.** Let X^H be the fixed point set of H in a G-manifold X with one orbit type $[H]$. For $\alpha \in \pi_0\left(X^H\right)$, let X_α^H denote the corresponding connected component of X^H.

Definition 5.2. *We denote $X_\alpha = GX_\alpha^H$, and X_α is called a **component of X relative to** G.*

Remark 5.3. *The space X_α is not necessarily connected, but it is the inverse image of a connected component of $G \backslash X$ under the projection $X \to G \backslash X$. Also, note that $X_\alpha = X_\beta$ if there exists n in the normalizer $N = N(H)$ such that $nX_\alpha^H = X_\beta^H$. If X is a closed manifold, then there are a finite number of components of X relative to G.*

We now introduce a decomposition of a G-bundle $E \to X$. Let E_α be the restriction $E|_{X_\alpha^H}$. For any irreducible representation $\sigma : H \to U(W_\sigma)$, we define for $n \in N$ the representation $\sigma^n : H \to U(W_\sigma)$ by $\sigma^n(h) = \sigma(n^{-1}hn)$. Let $\widetilde{N}_{[\sigma]} = \{n \in N : [\sigma^n] \text{ is equivalent to } [\sigma]\}$. If the isotypical component $E_\alpha^{[\sigma]}$ is nontrivial, then it is invariant under the subgroup $\widetilde{N}_{\alpha,[\sigma]} \subseteq \widetilde{N}_{[\sigma]}$ that leaves in addition the connected component X_α^H invariant; again, this subgroup has finite index in N. The isotypical components transform under $n \in N$ as

$$n : E_\alpha^{[\sigma]} \xrightarrow{\cong} E_{\overline{n}(\alpha)}^{[\sigma^n]} \ ,$$

where $\overline{n}$ denotes the residue class class of $n \in N$ in $N / \widetilde{N}_{\alpha,[\sigma]}$. Then a decomposition of E is obtained by 'inducing up' the isotypical components $E_\alpha^{[\sigma]}$ from $\widetilde{N}_{\alpha,[\sigma]}$ to N. That is,

$$E_{\alpha,[\sigma]}^N = N \times_{\widetilde{N}_{\alpha,[\sigma]}} E_\alpha^{[\sigma]}$$

is a bundle containing $E_\alpha^{[\sigma]}\Big|_{X_\alpha^H}$. This is an N-bundle over $NX_\alpha^H \subseteq X^H$, and a similar bundle may be formed over each distinct NX_β^H, with $\beta \in \pi_0\left(X^H\right)$. Further, observe that since each bundle $E_{\alpha,[\sigma]}^N$ is an N-bundle over NX_α^H, it defines a unique G bundle $E_{\alpha,[\sigma]}^G$ (see Exercise 75).

Definition 5.4. *The G-bundle $E_{\alpha,[\sigma]}^G$ over the submanifold X_α is called a **fine component** or the **fine component of $E \to X$ associated to** $(\alpha, [\sigma])$.*

If $G \backslash X$ is not connected, one must construct the fine components separately over each X_α. If E has finite rank, then E may be decomposed as a direct sum of distinct fine components

over each X_α. In any case, $E^N_{\alpha,[\sigma]}$ is a finite direct sum of isotypical components over each X^H_α.

Definition 5.5. *The direct sum decomposition of $E|_{X_\alpha}$ into subbundles E^b that are fine components $E^G_{\alpha,[\sigma]}$ for some $[\sigma]$, written*

$$E|_{X_\alpha} = \bigoplus_b E^b \ ,$$

*is called the **refined isotypical decomposition** (or **fine decomposition**) of $E|_{X_\alpha}$.*

We comment that if $[\sigma, W_\sigma]$ is an irreducible H-representation present in E_x with $x \in X^H_\alpha$, then $E^{[\sigma]}_x$ is a subspace of a distinct E^b_x for some b. The subspace E^b_x also contains $E^{[\sigma^n]}_x$ for every n such that $nX^H_\alpha = X^H_\alpha$.

Remark 5.6. *Observe that by construction, for $x \in X^H_\alpha$ the multiplicity and dimension of each $[\sigma]$ present in a specific E^b_x is independent of $[\sigma]$. Thus, $E^{[\sigma^n]}_x$ and $E^{[\sigma]}_x$ have the same multiplicity and dimension if $nX^H_\alpha = X^H_\alpha$.*

Remark 5.7. *The advantage of this decomposition over the isotypical decomposition is that each E^b is a G-bundle defined over all of X_α, and the isotypical decomposition may only be defined over X^H_α.*

Definition 5.8. *Now, let E be a G-equivariant vector bundle over X, and let E^b be a fine component as in Definition 5.4 corresponding to a specific component $X_\alpha = GX^H_\alpha$ of X relative to G. Suppose that another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E^b_y, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, and that W has a single component in the fine decomposition. Then we say that W is **adapted** to E^b.*

Lemma 5.9. *In the definition above, if another G-bundle W over X_α has finite rank and has the property that the equivalence classes of G_y-representations present in $E^b_y, y \in X_\alpha$ exactly coincide with the equivalence classes of G_y-representations present in W_y, then it follows that W has a single component in the fine decomposition and hence is adapted to E^b. Thus, the last phrase in the corresponding sentence in the above definition is superfluous.*

Exercise 75. *Suppose that X is a G-manifold, H is an isotropy subgroup, and $E' \to X^H$ is an $N(H)$-bundle over the fixed point set X^H. Prove that E' uniquely determines a G-bundle E over X such that $E|_{X^H} = E'$.*

Exercise 76. *Prove Lemma 5.9.*

5.5. **Canonical isotropy G-bundles.** In what follows, we show that there are naturally defined finite-dimensional vector bundles that are adapted to any fine components. Once and for all, we enumerate the irreducible representations $\left\{ \left[\rho_j, V_{\rho_j} \right] \right\}_{j=1,2,\ldots}$ of G. Let $[\sigma, W_\sigma]$ be any irreducible H-representation. Let $G \times_H W_\sigma$ be the corresponding homogeneous vector bundle over the homogeneous space G/H. Then the L^2-sections of this vector bundle decompose into irreducible G-representations. In particular, let $\left[\rho_{j_0}, V_{\rho_{j_0}} \right]$ be the equivalence class of irreducible representations that is present in $L^2\left(G/H, G \times_H W_\sigma \right)$ and that has the lowest index j_0. Then Frobenius reciprocity implies

$$0 \neq \mathrm{Hom}_G \left(V_{\rho_{j_0}}, L^2 \left(G/H, G \times_H W_\sigma \right) \right) \cong \mathrm{Hom}_H \left(V_{\mathrm{Res}\left(\rho_{j_0} \right)}, W_\sigma \right),$$

so that the restriction of ρ_{j_0} to H contains the H-representation $[\sigma]$. Now, for a component X_α^H of X^H, with $X_\alpha = GX_\alpha^H$ its component in X relative to G, the trivial bundle

$$X_\alpha \times V_{\rho_{j_0}}$$

is a G-bundle (with diagonal action) that contains a nontrivial fine component $W_{\alpha,[\sigma]}$ containing $X_\alpha^H \times \left(V_{\rho_{j_0}}\right)^{[\sigma]}$.

Definition 5.10. *We call $W_{\alpha,[\sigma]} \to X_\alpha$ the **canonical isotropy G-bundle associated to** $(\alpha, [\sigma]) \in \pi_0\left(X^H\right) \times \widehat{H}$. Observe that $W_{\alpha,[\sigma]}$ depends only on the enumeration of irreducible representations of G, the irreducible H-representation $[\sigma]$ and the component X_α^H. We also denote the following positive integers associated to $W_{\alpha,[\sigma]}$:*

- *$m_{\alpha,[\sigma]} = \dim \operatorname{Hom}_H\left(W_\sigma, W_{\alpha,[\sigma],x}\right) = \dim \operatorname{Hom}_H\left(W_\sigma, V_{\rho_{j_0}}\right)$ (the **associated multiplicity**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$ (see Remark 5.6).*
- *$d_{\alpha,[\sigma]} = \dim W_\sigma$(the **associated representation dimension**), independent of the choice of $[\sigma, W_\sigma]$ present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.*
- *$n_{\alpha,[\sigma]} = \frac{\operatorname{rank}\left(W_{\alpha,[\sigma]}\right)}{m_{\alpha,[\sigma]} d_{\alpha,[\sigma]}}$ (the **inequivalence number**), the number of inequivalent representations present in $W_{\alpha,[\sigma],x}$, $x \in X_\alpha^H$.*

Remark 5.11. *Observe that $W_{\alpha,[\sigma]} = W_{\alpha',[\sigma']}$ if $[\sigma'] = [\sigma^n]$ for some $n \in N$ such that $nX_\alpha^H = X_{\alpha'}^H$.*

Lemma 5.12. *Given any G-bundle $E \to X$ and any fine component E^b of E over some $X_\alpha = GX_\alpha^H$, there exists a canonical isotropy G-bundle $W_{\alpha,[\sigma]}$ adapted to $E^b \to X_\alpha$.*

Exercise 77. *Prove Lemma 5.12.*

Exercise 78. *Suppose G is a compact, connected Lie group, and T is a maximal torus. Let G act on left on the homogeneous space $X = G/T$.*

- *What is $(G/T)^T$?*
- *Let σ_a be a fixed irreducible representation of T (on $\mathbb{C}$), say $\sigma_a(t) = \exp(2\pi i (a \cdot t))$ with $a \in \mathbb{Z}^m$, $m = rank(T)$. Let $E = G \times_{\sigma_a} \mathbb{C} \to G/T$ be the associated line bundle. Is E a canonical isotropy G-bundle associated to $(\cdot, [\sigma_a])$?*
- *Is it true that every complex G-bundle over G/T is a direct sum of equivariant line bundles?*

5.6. **The equivariant index theorem.** To evaluate $\operatorname{ind}^\rho(D)$, we first perform the equivariant desingularization as described in Section 5.3, starting with a minimal stratum. In [12], we precisely determine the effect of the desingularization on the operators and bundles, and in turn the supertrace of the equivariant heat kernel. We obtain the following result. In what follows, if U denotes an open subset of a stratum of the action of G on M, U' denotes the equivariant desingularization of U, and $\widetilde{U}$ denotes the fundamental domain of U inside U', as in Section 5.3. We also refer the reader to Definitions 5.2 and 5.10.

Theorem 5.13. *(Equivariant Index Theorem, in [12]) Let M_0 be the principal stratum of the action of a compact Lie group G on the closed Riemannian M, and let $\Sigma_{\alpha_1},...,\Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. Let $E \to M$ be a Hermitian vector bundle on which G acts by isometries. Let $D : \Gamma(M, E^+) \to \Gamma(M, E^-)$ be a first order,*

transversally elliptic, G-equivariant differential operator. We assume that near each Σ_{α_j}, D is G-homotopic to the product $D_N * D^{\alpha_j}$, where D_N is a G-equivariant, first order differential operator on $B_\varepsilon \Sigma$ that is elliptic and has constant coefficients on the fibers and D^{α_j} is a global transversally elliptic, G-equivariant, first order operator on the Σ_{α_j}. In polar coordinates

$$D_N = Z_j \left(\nabla^E_{\partial_r} + \frac{1}{r} D^S_j \right) \ ,$$

where r is the distance from Σ_{α_j}, where Z_j is a local bundle isometry (dependent on the spherical parameter), the map D^S_j is a family of purely first order operators that differentiates in directions tangent to the unit normal bundle of Σ_j. Then the equivariant index $\mathrm{ind}^\rho(D)$ satisfies

$$\mathrm{ind}^\rho(D) = \int_{G \backslash \widetilde{M_0}} A^\rho_0(x) \ \widetilde{|dx|} + \sum_{j=1}^r \beta\left(\Sigma_{\alpha_j}\right) \ ,$$

$$\beta\left(\Sigma_{\alpha_j}\right) = \frac{1}{2 \dim V_\rho} \sum_{b \in B} \frac{1}{n_b \mathrm{rank}\, W^b} \left(-\eta\left(D^{S+,b}_j\right) + h\left(D^{S+,b}_j\right) \right) \int_{G \backslash \widetilde{\Sigma_{\alpha_j}}} A^\rho_{j,b}(x) \ \widetilde{|dx|} \ ,$$

where

(1) *$A^\rho_0(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from D' (blown-up and doubled from D) on the quotient M'_0 / G, where the bundle E is replaced by the finite rank bundle $\mathcal{E}_\rho$ of sections of type ρ over the fibers.*

(2) *Similarly, $A^\rho_{i,b}$ is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $(\mathbf{1} \otimes D^{\alpha_j})'$ (blown-up and doubled from $\mathbf{1} \otimes D^{\alpha_j}$, the twist of D^{α_j} by the canonical isotropy bundle $W^b \to \Sigma_{\alpha_j}$) on the quotient Σ'_{α_j} / G, where the bundle is replaced by the space of sections of type ρ over each orbit.*

(3) *$\eta\left(D^{S+,b}_j\right)$ is the eta invariant of the operator D^{S+}_j induced on any unit normal sphere $S_x \Sigma_{\alpha_j}$, restricted to sections of isotropy representation types in W^b_x; see [12]. This is constant on Σ_{α_j}.*

(4) *$h\left(D^{S+,b}_j\right)$ is the dimension of the kernel of $D^{S+,b}_j$, restricted to sections of isotropy representation types in W^b_x, again constant on on Σ_{α_j}.*

(5) *n_b is the number of different inequivalent G_x-representation types present in each W^b_x, $x \in \Sigma_{\alpha_j}$.*

As an example, we immediately apply the result to the de Rham operator and in doing so obtain an interesting equation involving the equivariant Euler characteristic. In what follows, let $\mathcal{L}_{N_j} \to \Sigma_j$ be the orientation line bundle of the normal bundle to the singular stratum Σ_j. The relative Euler characteristic is defined for X a closed subset of a manifold Y as $\chi(Y, X, \mathcal{V}) = \chi(Y, \mathcal{V}) - \chi(X, \mathcal{V})$, which is also the alternating sum of the dimensions of the relative de Rham cohomology groups with coefficients in a complex vector bundle $\mathcal{V} \to Y$. If $\mathcal{V}$ is an equivariant vector bundle, the **equivariant Euler characteristic** $\chi^\rho(Y, \mathcal{V})$ associated to the representation $\rho : G \to U(V_\rho)$ is the alternating sum

$$\chi^\rho(Y, \mathcal{V}) = \sum_j (-1)^j \dim H^j(Y, \mathcal{V})^\rho \ ,$$

where the superscript ρ refers to the restriction of these cohomology groups to forms of G-representation type $[\rho]$. An application of the equivariant index theorem yields the following result.

Theorem 5.14. *(Equivariant Euler Characteristic Theorem, in [12]) Let M be a compact G-manifold, with G a compact Lie group and principal isotropy subgroup H_{pr}. Let M_0 denote the principal stratum, and let $\Sigma_{\alpha_1},...,\Sigma_{\alpha_r}$ denote all the components of all singular strata relative to G. We use the notations for $\chi^\rho(Y,X)$ and $\chi^\rho(Y)$ as in the discussion above. Then*

$$\chi^\rho(M) = \chi^\rho(G/H_{\mathrm{pr}})\,\chi(G\backslash M, G\backslash singular\ strata)$$
$$+ \sum_j \chi^\rho\left(G/G_j\ ,\mathcal{L}_{N_j}\right)\chi\left(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash lower\ strata\right),$$

where $\mathcal{L}_{N_j}$ is the orientation line bundle of normal bundle of the stratum component Σ_{α_j}.

Exercise 79. *Let $M = S^n$, let $G = O(n)$ acting on latitude spheres (principal orbits, diffeomorphic to S^{n-1}). Show that there are two strata, with the singular strata being the two poles. Show without using the theorem by identifying the harmonic forms that*

$$\chi^\rho(S^n) = \begin{cases} (-1)^n & if\ \rho = \xi \\ 1 & if\ \rho = \mathbf{1} \end{cases},$$

where ξ is the induced one dimensional representation of $O(n)$ on the volume forms.

Exercise 80. *In the previous example, show that*

$$\chi^\rho(G/H_{\mathrm{pr}}) = \chi^\rho\left(S^{n-1}\right) = \begin{cases} (-1)^{n-1} & if\ \rho = \xi \\ 1 & if\ \rho = \mathbf{1} \end{cases},$$

and $\chi(G\backslash M, G\backslash singular\ strata) = -1$. Show that at each pole,

$$\chi^\rho\left(G/G_j, \mathcal{L}_{N_j}\right) = \chi^\rho(\mathrm{pt}) = \begin{cases} 1 & if\ \rho = \mathbf{1}, \\ 0 & otherwise. \end{cases}$$

and $\chi\left(G\backslash\overline{\Sigma_{\alpha_j}}, G\backslash lower\ strata\right) = 1$. Demonstrate that Theorem 5.14 produces the same result as in the previous exercise.

Exercise 81. *If instead the group $\mathbb{Z}_2$ acts on S^n by the antipodal map, prove that*

$$\chi^\rho(S^n) = \begin{cases} 0 & if\ \rho = \mathbf{1}\ or\ \xi\ and\ n\ is\ odd \\ 1 & if\ \rho = \mathbf{1}\ or\ \xi\ and\ n\ is\ even \\ 0 & otherwise \end{cases},$$

both by direct calculation and by using Theorem 5.14.

Exercise 82. *Consider the action of $\mathbb{Z}_4$ on the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, where the action is generated by a $\frac{\pi}{2}$ rotation. Explicitly, $k \in \mathbb{Z}_4$ acts on $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ by*

$$\phi(k)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Endow T^2 with the standard flat metric. Let ρ_j be the irreducible character defined by $k \in \mathbb{Z}_4 \mapsto e^{ikj\pi/2}$. Prove that

$$\chi^{\mathbf{1}}(T^2) = 2, \chi^{\rho_1}(T^2) = \chi^{\rho_3}(T^2) = -1, \chi^{\rho_2}(T^2) = 0,$$

in two different ways. First, compute the dimensions of the spaces of harmonic forms to determine the equations. Second, use the Equivariant Euler Characteristic Theorem.

5.7. **The basic index theorem for Riemannian foliations.** Suppose that E is a foliated Hermitian $\mathbb{C}l(Q)$ module with metric basic $\mathbb{C}l(Q)$ connection ∇^E over a Riemannian foliation $(M, \mathcal{F})$. Let

$$D_b^E : \Gamma_b\left(E^+\right) \to \Gamma_b\left(E^-\right)$$

be the associated basic Dirac operator, as explained in Section 3.1.

In the formulas below, any lower order terms that preserve the basic sections may be added without changing the index. Note that

Definition 5.15. *The* analytic basic index *of D_b^E is*

$$\mathrm{ind}_b\left(D_b^E\right) = \dim \ker D_b^E - \dim \ker \left(D_b^E\right)^*.$$

As shown explicitly in [13], these dimensions are finite, and it is possible to identify $\mathrm{ind}_b\left(D_b^E\right)$ with the invariant index of a first order, G-equivariant differential operator $\widehat{D}$ over a vector bundle over a basic manifold $\widehat{W}$, where G is $SO(q)$, $O(q)$, or the product of one of these with a unitary group $U(k)$. By applying the equivariant index theorem (Theorem 5.13) to the case of the trivial representation, we obtain the following formula for the index. In what follows, if U denotes an open subset of a stratum of $(M, \mathcal{F})$, U' denotes the desingularization of U very similar to that in Section 5.3, and $\widetilde{U}$ denotes the fundamental domain of U inside U'.

Theorem 5.16. *(Basic Index Theorem for Riemannian foliations, in [13]) Let M_0 be the principal stratum of the Riemannian foliation $(M, \mathcal{F})$, and let $M_1,$ ... , M_r denote all the components of all singular strata, corresponding to $O(q)$-isotropy types $[G_1],$ ... ,$[G_r]$ on the basic manifold. With notation as in the discussion above, we have*

$$\mathrm{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0}/\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^r \beta(M_j)$$

$$\beta(M_j) = \frac{1}{2}\sum_\tau \frac{1}{n_\tau \mathrm{rank}\ W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j}/\overline{\mathcal{F}}} A_{j,b}^\tau(x)\ \widetilde{|dx|},$$

where the sum is over all components of singular strata and over all canonical isotropy bundles W^τ, only a finite number of which yield nonzero $A_{j,b}^\tau$, and where

(1) $A_{0,b}(x)$ *is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\widetilde{D_b^E}$ (a desingularization of D_b^E) on the quotient $\widetilde{M_0}/\overline{\mathcal{F}}$, where the bundle E is replaced by the space of basic sections of over each leaf closure;*

(2) $\eta\left(D_j^{S+,b}\right)$ *and $h\left(D_j^{S+,b}\right)$ are defined in a similar way as in Theorem 5.13, using a decomposition $D_b^E = D_N * D_{M_j}$ at each singular stratum;*

(3) $A_{j,b}^\tau(x)$ *is the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $\left(\mathbf{1} \otimes D_{M_j}\right)'$ (blown-up and doubled from $\mathbf{1} \otimes D_{M_j}$, the twist of D_{M_j} by the canonical isotropy bundle W^τ) on the quotient $\widetilde{M_j}/\overline{\mathcal{F}}$, where the bundle is replaced by the space of basic sections over each leaf closure; and*

(4) n_τ is the number of different inequivalent G_j-representation types present in a typical fiber of W^τ.

An example of this result is the generalization of the Gauss-Bonnet Theorem to the basic Euler characteristic. Recall from Section 3.1 that the basic forms $\Omega(M, \mathcal{F})$ are preserved by the exterior derivative, and the resulting cohomology is called basic cohomology $H^*(M, \mathcal{F})$. The basic cohomology groups are finite-dimensional in the Riemannian foliation, and the basic Euler characteristic is defined to be

$$\chi(M, \mathcal{F}) = \sum (-1)^j \dim H^j(M, \mathcal{F}).$$

We have two independent proofs of the following Basic Gauss-Bonnet Theorem; one proof uses the result in [7], and the other proof is a direct consequence of the basic index theorem stated above (proved in [13]). We express the basic Euler characteristic in terms of the ordinary Euler characteristic, which in turn can be expressed in terms of an integral of curvature. We extend the Euler characteristic notation $\chi(Y)$ for Y any open (noncompact without boundary) or closed (compact without boundary) manifold to mean

$$\chi(Y) = \begin{array}{ll} \chi(Y) & \text{if } Y \text{ is closed} \\ \chi(\text{1-point compactification of } Y) - 1 & \text{if } Y \text{ is open} \end{array}$$

Also, if $\mathcal{L}$ is a foliated line bundle over a Riemannian foliation $(X, \mathcal{F})$, we define the basic Euler characteristic $\chi(X, \mathcal{F}, \mathcal{L})$ as before, using the basic cohomology groups with coefficients in the line bundle $\mathcal{L}$.

Theorem 5.17. *(Basic Gauss-Bonnet Theorem, announced in [49], proved in [13]) Let $(M, \mathcal{F})$ be a Riemannian foliation. Let $M_0, ..., M_r$ be the strata of the Riemannian foliation $(M, \mathcal{F})$, and let $\mathcal{O}_{M_j/\overline{\mathcal{F}}}$ denote the orientation line bundle of the normal bundle to $\overline{\mathcal{F}}$ in M_j. Let L_j denote a representative leaf closure in M_j. With notation as above, the basic Euler characteristic satisfies*

$$\chi(M, \mathcal{F}) = \sum_j \chi\left(M_j/\overline{\mathcal{F}}\right) \chi\left(L_j, \mathcal{F}, \mathcal{O}_{M_j/\overline{\mathcal{F}}}\right).$$

Remark 5.18. *In [23, Corollary 1], they show that in special cases the only term that appears is one corresponding to a most singular stratum.*

We now investigate some examples through exercises. The first example is a codimension 2 foliation on a 3-manifold. Here, $O(2)$ acts on the basic manifold, which is homeomorphic to a sphere. In this case, the principal orbits have isotropy type $(\{e\})$, and the two fixed points obviously have isotropy type $(O(2))$. In this example, the isotropy types correspond precisely to the infinitesimal holonomy groups.

Exercise 83. *(From [48], [52], and [13]) Consider the one dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere S^2. On S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1 - z^2)} \cos\theta$, $y' = \sqrt{(1 - z^2)} \sin\theta$, $z' = z$. Let α be an irrational multiple of 2π, and let the three–manifold $M = S^2 \times [0, 1]/\sim$, where $(z, \theta, 0) \sim (z, \theta + \alpha, 1)$. Endow M with the product metric on $T_{z,\theta,t}M \cong T_{z,\theta}S^2 \times T_t\mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta} = \cup_{n \in \mathbb{Z}} \{z\} \times \{\theta + \alpha\} \times [0, 1]$ (not unique in θ). The leaf closures $\overline{L}_z$ for $|z| < 1$ are two dimensional, and the closures corresponding to the poles $(z = \pm 1)$ are one dimensional.*

Show that $\chi(M, \mathcal{F}) = 2$, using a direct calculation of the basic cohomology groups and also by using the Basic Gauss-Bonnet Theorem.

The next example is a codimension 3 Riemannian foliation for which all of the infinitesimal holonomy groups are trivial; moreover, the leaves are all simply connected. There are leaf closures of codimension 2 and codimension 1. The codimension 1 leaf closures correspond to isotropy type (e) on the basic manifold, and the codimension 2 leaf closures correspond to an isotropy type $(O(2))$ on the basic manifold. In some sense, the isotropy type measures the holonomy of the leaf closure in this case.

Exercise 84. *(From [13]) This foliation is a suspension of an irrational rotation of S^1 composed with an irrational rotation of S^2 on the manifold $S^1 \times S^2$. As in Example 83, on S^2 we use the cylindrical coordinates (z, θ), related to the standard rectangular coordinates by $x' = \sqrt{(1 - z^2)} \cos \theta$, $y' = \sqrt{(1 - z^2)} \sin \theta$, $z' = z$. Let α be an irrational multiple of 2π, and let β be any irrational number. We consider the four–manifold $M = S^2 \times [0, 1] \times [0, 1] / \sim$, where $(z, \theta, 0, t) \sim (z, \theta, 1, t)$, $(z, \theta, s, 0) \sim (z, \theta + \alpha, s + \beta \mod 1, 1)$. Endow M with the product metric on $T_{z,\theta,s,t} M \cong T_{z,\theta} S^2 \times T_s \mathbb{R} \times T_t \mathbb{R}$. Let the foliation $\mathcal{F}$ be defined by the immersed submanifolds $L_{z,\theta,s} = \cup_{n \in \mathbb{Z}} \{z\} \times \{\theta + \alpha\} \times \{s + \beta\} \times [0, 1]$ (not unique in θ or s). The leaf closures $\overline{L}_z$ for $|z| < 1$ are three–dimensional, and the closures corresponding to the poles $(z = \pm 1)$ are two–dimensional. By computing the basic forms of all degrees, verify that the basic Euler characteristic is zero. Next, use the Basic Gauss-Bonnet Theorem to see the same result.*

The following example is a codimension two transversally oriented Riemannian foliation in which all the leaf closures have codimension one. The leaf closure foliation is not transversally orientable, and the basic manifold is a flat Klein bottle with an $O(2)$–action. The two leaf closures with $\mathbb{Z}_2$ holonomy correspond to the two orbits of type $(\mathbb{Z}_2)$, and the other orbits have trivial isotropy.

Exercise 85. *This foliation is the suspension of an irrational rotation of the flat torus and a $\mathbb{Z}_2$–action. Let X be any closed Riemannian manifold such that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$, the free group on two generators $\{\alpha, \beta\}$. We normalize the volume of X to be 1. Let $\widetilde{X}$ be the universal cover. We define $M = \widetilde{X} \times S^1 \times S^1 / \pi_1(X)$, where $\pi_1(X)$ acts by deck transformations on $\widetilde{X}$ and by $\alpha(\theta, \phi) = (2\pi - \theta, 2\pi - \phi)$ and $\beta(\theta, \phi) = (\theta, \phi + \sqrt{2}\pi)$ on $S^1 \times S^1$. We use the standard product–type metric. The leaves of $\mathcal{F}$ are defined to be sets of the form $\left\{(x, \theta, \phi)_\sim \mid x \in \widetilde{X}\right\}$. Note that the foliation is transversally oriented. Show that the basic Euler characteristic is 2, in two different ways.*

The following example (from [14]) is a codimension two Riemannian foliation that is not taut.

Exercise 86. *For the example in Exercise 44, show that the basic manifold is a torus, and the isotropy groups are all trivial. Verify that $\chi(M, \mathcal{F}) = 0$ in two different ways.*

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