Advanced Course: Transversal Dirac operators on distributions, foliations, and *G*-manifolds

Ken Richardson and coauthors

CENTRE DE RECERCA MATEMÀTICA

Universitat Autònoma de Barcelona

May 3-7, 2010

Curs avançat: estudi transversal sobre la distribució dels operadors de Dirac, foliacions i G-varietats

Ken Richardson i co-autors

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- **Jesús Álvarez López** (Universidade de Santiago de Compostela)
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First lecture: Transversal Dirac operators on Distributions

Ken Richardson
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May 3, 2010

This talk includes joint work with Igor Prohorenkov, *Natural Equivariant Dirac Operators*, to appear in Geom. Dedicata, preprint also available at arXiv:0805.3340v1 [math.DG]. Some of the photos are also done by Igor!

Plan of the talk

- **Introduction to ordinary Dirac operators**
  - Simple example
  - General Definition
  - Examples and Properties
  - Atiyah-Singer Index Theorem
- **Transversal Dirac Operators on Distributions**
  - General definition
  - Discussion of mean curvature
  - Example: Foliations
  - Restricting Spaces of Sections
- **Goals of next lectures**

The ordinary Dirac-type operator

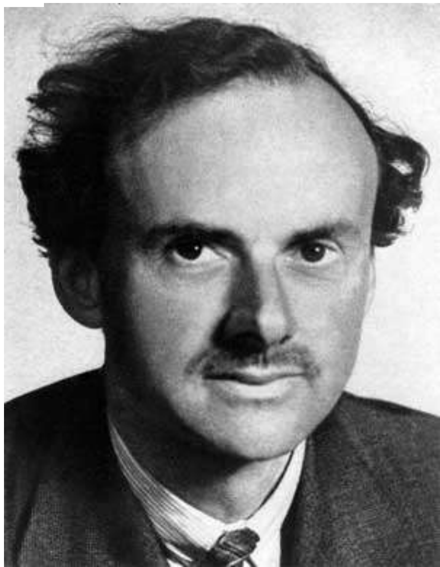
- Euclidean space:

$$D=\sum e_i\cdot\frac{\partial}{\partial x_i},\quad e_i\cdot e_j\cdot+e_j\cdot e_i\cdot \;=-2\delta_{ij}\mathbf{1}$$

- 2 dimensions:

$$D=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\frac{\partial}{\partial y}$$

- In general: $D=\sum c(e_j)\nabla_j^E$
- Physics: eigensections correspond to particles with spin at different energy levels.



General,Coordinate-free version

- $E = E^+ \oplus E^-$ is a graded, self-adjoint Clifford module over a closed, Riemannian manifold M with compatible connection ∇^E. Let $\Gamma(M,E)$ denote the space of smooth sections. Let $c : TM \to \mathrm{End}^-(E)$ be Clifford multiplication.
- Dirac operator $D : \Gamma(M,E) \to \Gamma(M,E)$ is the composition

$$\Gamma(E) \overset{\nabla^E}{\to} \Gamma(T^*M \otimes E) \overset{\cong}{\to} \Gamma(TM \otimes E) \overset{c}{\to} \Gamma(E)$$

Examples of Dirac operators

- "The" spin-c Dirac operator
- The de Rham operator

$$d+\delta\ :\Omega^{\text{even}}(M)\to\Omega^{\text{odd}}(M)$$

- The signature operator

$$d+\delta\ :\Omega^{+}(M)\to\Omega^{-}(M)$$

Here, $\bigstar=\pm *,\ \bigstar^2=\mathbf{1}$

$$\Omega^{\pm}(M)=\{\alpha\in\Omega^{*}(M)\ :\bigstar\alpha=\pm\alpha\}$$

- The Dolbeault operator

$$\partial+\overline{\partial}\ :\Omega^{0,\text{even}}(M)\to\Omega^{0,\text{odd}}(M)$$

More on the de Rham operator

- The Laplacian is $(d+\delta)^2 = \Delta$
- We have
$$\ker(d+\delta) = \{\text{harmonic forms}\}$$
- And ….
$$\operatorname{index}(d+\delta)|_{\Omega^{\text{even}}}$$
$$= \dim\ker(d+\delta)|_{\Omega^{\text{even}}} - \dim\ker\left((d+\delta)|_{\Omega^{\text{even}}}\right)^*$$
$$= \chi(M)$$

Properties of Dirac Operators

- They are elliptic
  - Differentiate in all directions (principal symbol matrix is invertible on each nonzero cotangent vector)
  - Eigensections are smooth (elliptic regularity)
  - Various inequalities hold
- They are essentially self-adjoint
- Important in Bott periodicity, Thom isomorphism in K-theory, and index theory, in that Dirac operator symbols generate all possible classes of symbols.

Index Theory

Let M be a smooth, closed manifold, and let

$$D : \Gamma(M, E) \to \Gamma(M, E)$$

be an elliptic operator.

Index(D) := dim ker D – dim coker D
= dim ker D – dim ker D^*
(in presence of metrics)

***Atiyah-Singer Index Thm*:**

$$\text{Index}(D) = \int_M \text{AS}$$

Examples

$$\text{Index}\left(d+d^* \big/_{\text{even}\to\text{odd}}\right) = \chi(M) = \int_M K/_{2\pi}$$

$$\text{Index}\left(d+d^* \big/_{+\to-}\right) = \text{Sign}(M) = \int_M L$$

$$\text{Index}\left(\partial^+\right) = \int_M \hat{A}$$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Dirac operators associated to Distributions

- A **distribution** $Q \subset TM$ is a smooth subbundle of the tangent bundle. All other notation as before.
- But $c : Q \to \mathrm{End}(F)$ is the Clifford action that will make E into a module. Let $(f_1, \ldots, f_q)$ be a local o-n frame for $Q \subset TM$. Projection: $\pi : TM \to Q$
- Define $A_Q = \sum_{j=1}^{q} c(f_j)\nabla_{f_j}^{E}$

Dirac operators associated to Distributions

- The operator $A_Q = \sum_{j=1}^{q} c(f_j)\nabla_{f_j}^{E}$ is independent of choices made. It is the composition

$$\Gamma(E) \stackrel{\nabla^E}{\to} \Gamma(T^*M \otimes E)$$

$$\stackrel{\cong}{\to} \Gamma(TM \otimes E) \stackrel{\pi}{\to} \Gamma(Q \otimes E) \stackrel{c}{\to} \Gamma(E)$$

- But: it's not elliptic, and not self-adjoint.

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Formal adjoint of A_Q

$$(A_Q s_1, s_2) = \sum_{j=1}^{n} \left(c(\pi f_j) \nabla^E_{f_j} s_1, s_2 \right)$$

$$= \sum -\left(\nabla^E_{f_j} s_1, c(\pi f_j) s_2 \right)$$

$$= \sum \left(-f_j (s_1, c(\pi f_j) s_2) + \left(s_1, \nabla^E_{f_j} c(\pi f_j) s_2 \right) \right)$$

$$= \sum \Big(- f_j (s_1, c(\pi f_j) s_2) + \left(s_1, c(\pi f_j) \nabla^E_{f_j} s_2 \right)$$

$$+ \left(s_1, c\left(\pi \nabla^M_{f_j} \pi f_j \right) s_2 \right) \Big)$$

$$(A_Q s_1, s_2) = \delta\omega - (s_1, c(V + H^L) s_2)$$

$$+ (s_1, A_Q s_2) + (s_1, c(V) s_2)$$

$$(A_Q s_1, s_2) = \delta\omega + (s_1, A_Q s_2) - (s_1, c(H^L) s_2)$$

Formal adjoint of A_Q, continued

$$(A_Q s_1, s_2) = \delta\omega + (s_1, A_Q s_2)$$
$$- (s_1, c(V + H^L - V) s_2)$$

$$V = \sum_{j=1}^{q} \pi \nabla^M_{f_j} f_j \ , \ H^L = \sum_{j=q+1}^{n} \pi \nabla^M_{f_j} f_j$$

Thus, $A_Q^* = A_Q - c(H^L)$

And.... $D_Q = A_Q - \frac{1}{2} c(H^L)$ **is s.-adjoint**

Second lecture: Basic Dirac operators on Riemannian Foliations

Ken Richardson
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May 4, 2010

•(with Igor Prohorenkov) *Natural Equivariant Dirac Operators*, to appear in Geom. Dedicata.

•(with Georges Habib) *A brief note on the spectrum of the basic Dirac operator*, Bull. London Math. Soc. 41(2009), 683-690.

•(with Georges Habib) *Modified differentials and basic cohomology for Riemannian foliations*, preprint in preparation.

•Note: all papers and slides available on my website: Google “Ken Richardson”
(slides from this conference to be posted later)

Plan of the talk

- **Elementary Foliation Concepts**
  - Smooth foliations
  - Riemannian foliations
  - Bundle-like metrics
- **Basic Dirac operator construction**
  - Recall transversal Dirac Operator with respect to a smooth distribution and adjoint calculation
  - Basic Dirac operator
  - Invariance of the spectrum of the basic Dirac operator

Smooth foliations

- Let $(M, \mathcal{F})$ be a smooth manifold along with a smooth foliation $\mathcal{F}$. This means that $\mathcal{F}$ is a partition of M into immersed submanifolds (*leaves*) such that the transition functions for the local product neighborhoods are smooth.
- Let $T\mathcal{F}$ denote the tangent bundle to the foliation. At each point p of M, $T_p\mathcal{F}$ is the tangent space to the leaf through p.

Basic forms

- Basic forms of $(M, \mathcal{F})$ are forms that are smooth on M but that locally are forms which depend only on the variables of the leaf spaces, the quotient of the local product by the leaves. Let $\Omega(M,F) \subset \Omega(M)$ denote the space of basic forms.
- Invariant definition:

$$\Omega(M,F) = \{\, \beta \in \Omega(M) \ : i(X)\beta = 0,$$

$$i(X)\, d\beta = 0, \ \forall X \in TF \,\}$$

Riemannian foliations

- Riemannian foliations are smooth foliations endowed with a holonomv-invariant metric on the quotient bundle $Q = TM/T\mathcal{F}$.
- This means that Q has a metric g_Q such that

$$\mathcal{L}_X g_Q = 0, \ \forall X \in \Gamma(T\mathcal{F})$$

- ($\mathcal{L}_X$ means Lie derivative with respect to X.)
- One thinks of this as a metric on the (singular) space of leaves. But not really.

Additional structure: bundle-like metrics

- A ***bundle-like metric*** on a Riemannian foliation $\left(M, \mathcal{F}, g_Q\right)$ is a metric on M such that its restriction to $N\mathcal{F} = (T\mathcal{F})^{\perp}$ agrees with g_Q through the natural isomorphism $Q \simeq N\mathcal{F}$.
- Every Riemannian foliation admits bundle-like metrics, and there are many choices of bundle-like metrics that are compatible with a given $\left(M, \mathcal{F}, g_Q\right)$ structure. One may freely choose the metric on the leaves and also the transverse subbundle $N\mathcal{F}$.

Bundle-like metrics, continued

- A bundle-like metric on a smooth foliation is exactly a metric on the manifold such that the leaves of the foliation are locally equidistant.
- There are topological restrictions to the existence of bundle-like metrics (and thus Riemannian foliations). Important examples include:
  - The leaf closures partition the manifold(are disjoint).
  - The basic cohomology is finite-dimensional.
  - The orthogonal projection $P : L^2(\Omega(M)) \to L^2(\Omega(M, \mathcal{F}))$ maps the subspace of smooth forms onto the subspace of smooth basic forms.

Riemannian foliations, continued

Historical note:

Riemannian foliations were introduced by B. Reinhart in 1959. Good references for Riemannian foliations and bundle-like metrics include the books and papers by B. Reinhart, F. Kamber, Ph. Tondeur, P. Molino, etc.

The mean curvature form

- Let $H = \sum_{j=1}^{p} \pi \nabla^M_{e_j} e_j$, where $\pi : TM \to N\mathcal{F}$ is the bundle projection, and $\{e_j\}_{j=1}^{p}$ is a local orthonormal frame of $T\mathcal{F}$. This is the mean curvature vector field, and its dual one-form is $\kappa = H^\flat$. The projection $\kappa_b = P\kappa$ is the basic projection of this one-form.
- Let $\kappa_b^\#$ be the corresponding (basic) vector field.
- κ_b is a closed form whose cohomology class in $H^1(M, \mathcal{F})$ is independent of the choice of bundle-like metric (see Álvarez-López )

Dirac operators associated to Distributions

- The operator $A_Q = \sum_{j=1}^{q} c(f_j)\nabla_{f_j}^{E}$ is independent of choices made. It is the composition

$$\Gamma(E) \overset{\nabla^E}{\to} \Gamma(T^*M \otimes E)$$

$$\overset{\cong}{\to} \Gamma(TM \otimes E) \overset{\pi}{\to} \Gamma(Q \otimes E) \overset{c}{\to} \Gamma(E)$$

- But: it’s not elliptic, and not self-adjoint.

Recall: Formal adjoint of A_Q

$$(A_Q s_1, s_2) = \sum_{j=1}^{n} \left(c(\pi f_j) \nabla^E_{f_j} s_1, s_2 \right)$$

$$= \sum -\left(\nabla^E_{f_j} s_1, c(\pi f_j) s_2 \right)$$

$$= \sum \left(-f_j (s_1, c(\pi f_j) s_2) + \left(s_1, \nabla^E_{f_j} c(\pi f_j) s_2 \right) \right)$$

$$= \sum \Big(- f_j (s_1, c(\pi f_j) s_2) + \left(s_1, c(\pi f_j) \nabla^E_{f_j} s_2 \right)$$

$$+ \left(s_1, c\left(\pi \nabla^M_{f_j} \pi f_j \right) s_2 \right) \Big)$$

$$(A_Q s_1, s_2) = \delta\omega - (s_1, c(V + H^L) s_2)$$

$$+ (s_1, A_Q s_2) + (s_1, c(V) s_2)$$

$$(A_Q s_1, s_2) = \delta\omega + (s_1, A_Q s_2) - (s_1, c(H^L) s_2)$$

Formal adjoint of A_Q, continued

$$(A_Q s_1, s_2) = \delta\omega + (s_1, A_Q s_2)$$
$$- (s_1, c(V + H^L - V)s_2)$$

$$V = \sum_{j=1}^{q} \pi\nabla^M_{f_j} f_j \ , \ H^L = \sum_{j=q+1}^{n} \pi\nabla^M_{f_j} f_j$$

Thus, $A_Q^* = A_Q - c(H^L)$

And…. $D_Q = A_Q - \frac{1}{2}c(H^L)$ **is s.-adjoint**

Basic Dirac operators

- Given a Riemannian foliation $(M, \mathcal{F})$ of codim q with compatible bundle-like metric, E a foliated $\mathbb{C}\mathrm{l}(Q)$ -module , the orthogonal projection $P_b : L^2(\Gamma(E)) \to L^2(\Gamma_b(E))$

- Let $D_b := P_b D_{N\mathcal{F}} P_b$

$$= A_{N\mathcal{F}} - \frac{1}{2} c\left(\kappa_b^\sharp\right) : \Gamma_b(E) \to \Gamma_b(E)$$

- References: Glazebrook-Kamber, S.D.Jung, Brüning-Kamber-R




Additional Structure Needed

- The connection on the vector bundle is basic, meaning the connection is flat along the leaves.
- The basic sections are defined to be

$$\Gamma_b(E) = \{s \in \Gamma(E) : \nabla_X^E s = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\}$$

- Let $P_b : L^2(\Gamma(E)) \to L^2(\Gamma_b(E))$ be the orthogonal projection (called the ***basic projection***). Then the basic projection preserves the smooth sections (Park-R, AJM paper). One can show:

$$P_b A_{N\mathcal{F}} P_b = A_{N\mathcal{F}} P_b,\ P_b c(\kappa^\sharp) P_b = c\left(\kappa_b^\sharp\right) P_b$$

Example: basic de Rham operator

- We have

$$D_b = d + \delta_b - \frac{1}{2}\kappa_b \lrcorner - \frac{1}{2}\kappa_b \wedge.$$

$$= \widetilde{d} + \widetilde{\delta}.$$

$$\widetilde{d} = d - \tfrac{1}{2}\kappa_b \wedge, \widetilde{\delta} = \delta_b - \tfrac{1}{2}\kappa_b \lrcorner$$

The spectrum of a basic Dirac operator

Theorem (Habib-R) Fix a Riemannian foliation. For any bundle-like metric and basic Dirac operator compatible with the Riemannian structure, the spectrum of this basic Dirac operator is independent of the choice of bundle-like metric.

Idea of proof

- One can show that every different choice of bundle-like metric changes the L^2-inner product by multiplication by a specific smooth, positive basic function.
- This changes the basic Dirac operator by a zeroth order operator that is Clifford multiplication by an exact basic one-form.
- This new operator is conjugate to the original one.

Physics Intuition

- If one pretends that points of space are actually immersed submanifolds of a larger space (This is the case with string theory),

$$D_b s = \sum_{i=1}^{q} e_i \bullet \nabla_{e_i}^{E} s - \frac{1}{2}\kappa_b^{\sharp} \bullet s$$

  is the corresponding Dirac operator.
- Note the presence of the term $\frac{1}{2}\kappa_b^{\#} \bullet$ is equivalent to the presence of a background field (whose vector potential is that vector field).

Consequences

- D. Dominguez showed that every Riemannian foliation admits a bundle-like metric for which the mean curvature form is basic.
- Further, March/MinOo/Ruh and A. Mason showed that the bundle-like metric may be chosen so that the mean curvature is basic-harmonic for that metric.
- Therefore, in calculating or estimating the eigenvalues of the basic Dirac operator, one may choose the bundle-like metric so that the mean curvature is basic-harmonic.
- Immediately we may obtain stronger inequalities for eigenvalue estimates.

- (with Georges Habib) ***Modified differentials and basic cohomology for Riemannian foliations***, **preprint in preparation.**

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Basic cohomology

- Note that the exterior derivative ***d*** maps basic forms to basic forms, and thus we may define the basic cohomology as

$$H^k(M,F) = \frac{\ker d_k}{\operatorname{Im} d_{k-1}}$$

with

$$d_k = d : \Omega^k(M,F) \to \Omega^{k+1}(M,F)$$

- Physics-y interpretation: we are doing calculus with the set of leaves (strings?). One can think of basic forms as forms on the space of leaves (a singular, possibly non-Hausdorff space) or of leaf closures, and basic cohomology is a smooth version of the cohomology of that space. But not really.

Basic cohomology, continued

- Basic cohomology can be infinite dimensional. It can be really trivial.
- We can also define basic cohomology of forms twisted by a vector bundle. If one does this, one can learn a lot about the topology of the leaf space of the foliation.
- Basic Cohomology does not satisfy Poincaré duality. If there are additional restrictions, it may satisfy duality --- for example, if the manifold has a metric for which the leaves are equidistant and are minimal submanifolds.

The basic Laplacian

- On a Riemannian foliation with bundle-like metric, one can define a basic Laplacian that maps basic forms to basic forms:
  $$\Delta_b = d\delta_b + \delta_b d : \Omega(M,F) \to \Omega(M,F)$$
  where δ_b is the L^2-adjoint of the restriction of *d* to basic forms: $\delta_b = P\delta$ is the ordinary adjoint of *d* followed by the orthogonal projection to the space of basic forms.
- This operator and its spectrum depend on the choice of bundle-like metric.

The basic adjoint δ_b

Properties of the adjoint δ_b of the restriction of ***d*** to basic forms (*see Park-R paper in Amer J Math*):

$$\begin{aligned} \delta_b &= P\delta \\ &= \pm\overline{*}d\overline{*} + \kappa_b \lrcorner \\ &= \delta_T + \kappa_b \lrcorner, \end{aligned}$$

Here, $\overline{*}$ is the pointwise transversal Hodge star operator, and $\kappa_b \lrcorner$ denotes interior product with the projection of the mean curvature one-form onto basic forms, and δ_T is the formal adjoint of ***d*** on the local transversals.

Recall: the mean curvature form

- Let $H = \sum_{j=1}^{p} \pi \nabla^M_{e_j} e_j$, where $\pi : TM \to N\mathcal{F}$ is the bundle projection, and $\{e_j\}_{j=1}^{p}$ is a local orthonormal frame of $T\mathcal{F}$. This is the mean curvature vector field, and its dual one-form is $\kappa = H^\flat$. The projection $\kappa_b = P\kappa$ is the basic projection of this one-form.
- Let $\kappa_b^\sharp$ be the corresponding (basic) vector field.
- κ_b is a closed form whose cohomology class in $H^1(M, \mathcal{F})$ is independent of the choice of bundle-like metric (see Álvarez-López )

Twisted duality for basic cohomology

- Even for Riemannian foliations, Poincaré duality does not hold for basic cohomology.
- However, note that $d - \kappa_b \wedge$ is also a differential which defines a cohomology of basic forms, and $\overline{*}\delta_b = \pm(d - \kappa_b \wedge)\overline{*}$, etc.
- Conclusion (Kamber/Tondeur/Park/R): if *q* is the codimension of the Riemannian foliation, then $H_d^*(M,F) \cong H_{d-\kappa_b\wedge}^{q-*}(M,F)$

Basic Laplacian, continued

- The basic Laplacian can be used to define transversal heat flow on Riemannian foliations with bundle-like metrics. Such a flow corresponds to assuming that the leaves of the foliation are perfect conductors of heat.
- It turns out that the basic Laplacian is the restriction of a non-symmetric second order elliptic operator on all forms. Only in special cases is it the same as the ordinary Laplacian.
- It is also not the same as the formal Laplacian defined on the local quotient (or transversal). This transversal Laplacian is in general not symmetric on the space of basic forms, but it does preserve the basic forms.
- The basic heat flow asymptotics are more complicated than that of the standard heat kernel, but there is a fair amount known (Transactions AMS paper to appear).

Spectrum of the Basic Laplacian

- Besides the basic heat equation asymptotics, there are some results estimating the eigenvalues of the basic Laplacian in terms of the Riemannian foliation geometry. These results can be found in the work of Lee-R, Jung-R., Jung, Habib
- For example, if the normal Ricci curvature of a Riemannian foliation of codimension q satisfies $Ric^{\perp}(X,X) \geq a(q-1)|X|^2$ for all $X \in NF$, then the first eigenvalue of Δ_b satisfies $\lambda \geq aq$. (Lee – R, 2002) *Interesting:* bound independent of choice of bundle-like metric.

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Reminder: Consequences

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- Therefore, in calculating or estimating the eigenvalues of the basic Dirac operator, one may choose the bundle-like metric so that the mean curvature is basic-harmonic.
- Immediately we may obtain stronger inequalities for eigenvalue estimates.

A modified differential

- Recall the basic de Rham operator is

$$D_b = d + \delta_b - \frac{1}{2}\kappa_b \lrcorner - \frac{1}{2}\kappa_b \wedge$$

$$= \widetilde{d} + \widetilde{\delta}$$

  where $\widetilde{d} = d - \frac{1}{2}\kappa_b \wedge$, $\widetilde{\delta} = \delta_b - \frac{1}{2}\kappa_b \lrcorner$
- These are differentials and codifferentials on basic forms.
- Bonus: $\widetilde{\delta}\overline{*} = \pm\overline{*}\widetilde{d}$ … etc.
- Thus the basic signature operator can be defined!

New basic Laplacian and basic cohomology

- Let $\widetilde{\Delta} = \left(\widetilde{d} + \widetilde{\delta}\right)^2$. Then the eigenvalues of this operator depend only on the Riemannian foliation structure.
- The new basic cohomology satisfies Poincaré duality and basic Hodge theory, and the isomorphism is:

$$\overline{*} \ : H^*_{\widetilde{d}}(M,F) \xrightarrow{\cong} H^{q-*}_{\widetilde{d}}(M,F)$$

- Even though the differential depends on the choice of bundle-like metric, the dimensions of the cohomology groups are independent of that choice.


Thank you for your
attention….