Differential Forms & Surface Integrals.

in $\mathbb{R}^3$

2 Form

$dx \wedge dz$ ← Think of this as

$\Delta x \Delta z$

also with
a vector direction

In $\mathbb{R}^3$ — think of as a vector $\perp$ to both $\vec{\Delta x}$ and $\vec{\Delta z}$ with direction given by the right hand rule (i.e. cross product)

1-Form dy -

(other times

)

Surface

M is a 2-d surface

$$\int_M A\, \underbrace{dx \wedge dy}_{z \text{ direction}} + B\, \underbrace{dy \wedge dz}_{x \text{ direction}} + C\, \underbrace{dx \wedge dz}_{y \text{ direction}}$$

↑ What does this mean?

$(B, -C, A) \cdot$ normal to surface $\cdot$ piece of area

→ Flux of the vector field $\langle B, -C, A\rangle$ through the surface M.

To actually compute such an integral, you need a parametrization of M

$$\phi(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$$

2 parameters (surface)

$$\int_M A^{(x,y,z)}\, dx\wedge dy + B^{(x,y,z)}\, dy\wedge dz + C^{(x,y,z)}\, dx\wedge dz$$

$F(u,v)\, du\wedge dv$

$$= \underbrace{\iint_{\substack{u,v \\ \text{parameter space}}} A\big(x(u,v), y(u,v), z(u,v)\big)\, d\big(x(u,v)\big)\wedge d\big(y(u,v)\big) + B \sim\sim + C \sim\sim}_{\text{Double integral in the } (u,v) \text{ plane.}}$$

$$(x_u\, du + x_v\, dv)\wedge(y_u\, du + y_v\, dv)$$

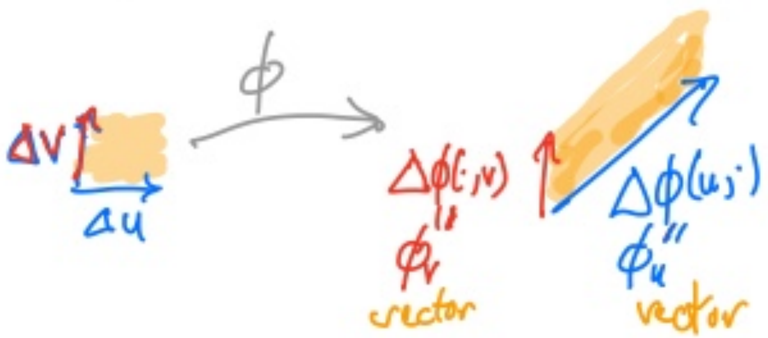
Sometimes we want to do an integral of a scalar function over a surface

$$\int_M \underbrace{F(x,y,z)}_{\text{Real-valued function}} \underbrace{dS}_{\substack{dA \\ \text{piece of surface area.}}}$$

eg: $\int_M 1 \, dS$ = surface area of M.

To do this: we still need a parametrization

$$\phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$



$V \cdot W = |v||w|\cos\theta$

$|V \times W| = |v||w|\sin\theta$ = area of parallelogram.

∴ little piece of area

$$= |\phi_u \times \phi_v| \, du \, dv$$

$$(x_u, y_u, z_u) \times (x_v, y_v, z_v)$$

$$\int_M F(x,y,z)\,dS = \iint_{u,v} \underbrace{F(x(u,v),y(u,v),z(u,v))\,|\phi_u \times \phi_v|\,du\,dv}_{\text{double integral}}$$

u,v parameter space

<u>Example:</u> Consider the 2-d torus in $\mathbb{R}^3$

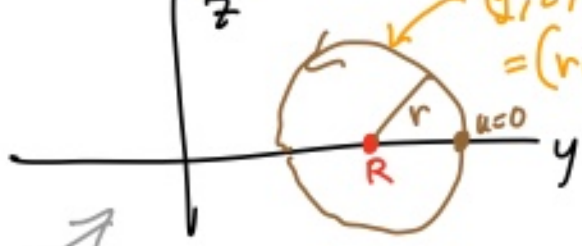
R = distance from center to middle of donut

r = distance from middle of donut to the icing of the donut.

① Let's parametrize it, place it in $\mathbb{R}^3$.

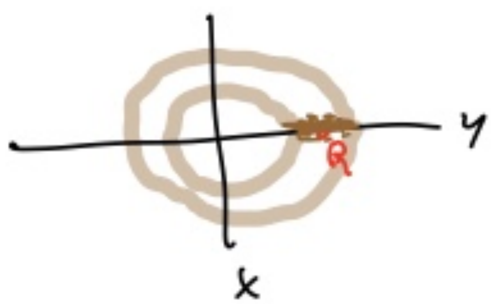
yz plane slice circle

$(y,z) = \cancel{(r\cos(u), r\sin(u))}$
$= (r\cos(u) + R, r\sin(u))$



Now we need to wind around xy plane

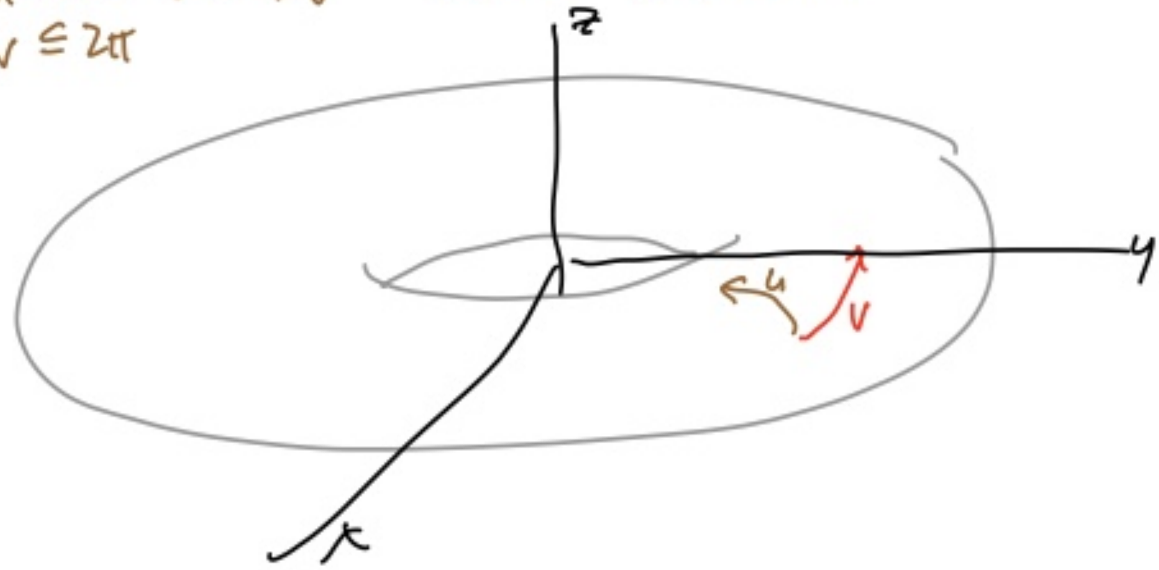
Top view (z upward)



$\phi(u,v) = \left((r\cos(u)+R)\cos(v), (r\cos(u)+R)\sin(v), r\sin(u) \right)$

$0 \leq u \leq 2\pi$ $\quad \phi_u \times \phi_v$ inward normal vector to surface

$0 \leq v \leq 2\pi$



Let's compute the surface area of this torus.

$$\phi(u,v) = \Big(\overbrace{(r\cos(u)+R)\cos(v)}^{x(u,v)}, \overbrace{(r\cos(u)+R)\sin(v)}^{y(u,v)}, \overbrace{r\sin(u)}^{z(u,v)} \Big)$$

$$\Rightarrow \phi_u = \left(-r\sin(u)\cos(v), -r\sin(u)\sin(v), r\cos(u) \right)$$

$$\phi_v = \left(-(r\cos(u)+R)\sin(v), (r\cos(u)+R)\cos(v), 0 \right)$$

$$\phi_u \times \phi_v = \begin{vmatrix} i & j & k \\ -r\sin(u)\cos(v) & -r\sin(u)\sin(v) & r\cos(u) \\ -(r\cos(u)+R)\sin(v) & (r\cos(u)+R)\cos(v) & 0 \end{vmatrix}$$

$$= i\left(-r\cos(u)(r\cos(u)+R)\cos v \right)$$

$$- j\left(+r\cos(u)(r\cos(u)+R)\sin(v) \right)$$

$$+ k\left(-r\sin(u)\cos(v)(r\cos(u)+R)\cos(v) - r\sin(u)\sin(v) \cdot (r\cos(u)+R)\sin(v) \right)$$

$$= i\left(-r\cos(u)\cos(v)(r\cos(u)+R)\right)$$
$$+j\left(-r\cos(u)\sin(v)(r\cos(u)+R)\right)$$
$$+k\left(-r\sin(u)(r\cos(u)+R)\right)$$

$$\Rightarrow |\phi_u \times \phi_v|^2 = (r\cos(u)+R)^2 r^2 \left(\cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u)\right)$$

$$\Rightarrow |\phi_u \times \phi_v| = r(r\cos(u)+R).$$

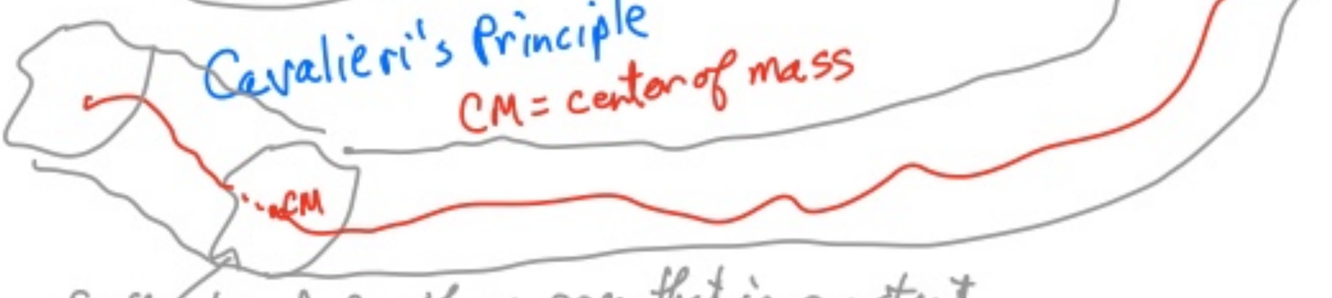
Surface Area of Torus

$$= \int_M 1\, dS = \int_0^{2\pi}\int_0^{2\pi} 1\cdot|\phi_u \times \phi_v|\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \underbrace{r(r\cos(u)+R)}_{(r^2\cos(u)+rR)}\, du\, dv$$

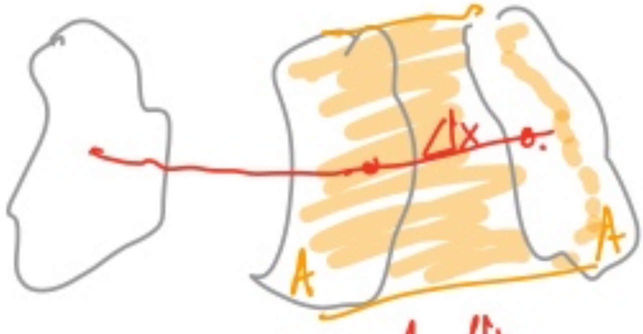
$$= \int_0^{2\pi} rR(2\pi)\, dv = \boxed{4\pi^2 rR}.$$

Another way: $(2\pi r)(2\pi R) = 4\pi^2 r R$



Surface area = (length of cross-section curve) · (length of CM curve)

Volume inside = (area of cross-section) · (length of CM curve).



$\Delta V = A\,\Delta x$