# UNBOUNDED OPERATORS ON HILBERT SPACES

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Let $X$ and $Y$ be normed linear spaces, and suppose $A : X \longrightarrow Y$ is a linear map. Define

$$\|A\|_{op} = \sup\left\{\frac{\|Ax\|}{\|x\|} : x \neq 0\right\} = \{\|Ax\| : \|x\| \leq 1\} = \{\|Ax\| : \|x\| = 1\}$$

If $\|A\| < \infty$, we call $A$ a *bounded* (linear) operator.

**Theorem 1.** *The following are equivalent:*

(a) *$A$ is bounded;*
(b) *$A$ is continuous at $0$;*
(c) *$A$ is uniformly continuous.*

Many commonly occurring operators are bounded; here is an example that is unbounded.

**Example 2.** *Let $\mathcal{B}$ be a vector space basis (or a* Hamel *basis, as it is often called) of $L^2(\mathbb{R})$ and choose a sequence $\{f_n\}$ of distinct elements of $\mathcal{B}$. By dividing each $f_n$ by its norm if necessary, we may assume that each $f_n$ has norm $1$. Define a linear map $F : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ by setting $Ff_n = n$ for each $n$ and $Fg = 0$ for all other elements in $\mathcal{B}$. Then $F$ is unbounded.*

From now on, we restrict our attention to linear operators from a Hilbert space $\mathcal{H}$ to itself. If $A : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded linear map, its *adjoint* $A^* : \mathcal{H} \longrightarrow \mathcal{H}$ is determined by the requirement that $\langle Af, g\rangle = \langle f, A^*g\rangle$ for all $f$ and $g$ in $\mathcal{H}$. Interestingly, the existence of an adjoint automatically implies that $A$ is linear.

**Theorem 3.** *Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a function, and suppose there exists a function $B : \mathcal{H} \longrightarrow \mathcal{H}$ with the property that $\langle Af, g\rangle = \langle f, Bg\rangle$ for all $f$ and $g$ in $\mathcal{H}$. Then $A$ is linear.*

*Proof.* For all $f$, $g$, and $h$ in $\mathcal{H}$ and complex numbers $\alpha$ and $\beta$,

$$\begin{aligned}
\langle A(\alpha f + \beta g), h\rangle &= \langle \alpha f + \beta g, Bh\rangle \\
&= \alpha\langle f, Bh\rangle + \beta\langle g, Bh\rangle \\
&= \alpha\langle Af, h\rangle + \beta\langle Ag, h\rangle \\
&= \langle \alpha Af + \beta Ag, h\rangle,
\end{aligned}$$

whence $A(\alpha f + \beta g) = \alpha Af + \beta Ag$. □

In fact, more is true.

**Theorem 4** (Closed Graph Theorem)**.** *Let $X$ and $Y$ be Banach spaces and suppose $A : X \longrightarrow Y$ is a linear map with the following property: if $\{x_n\}$ is a sequence in $X$ converging to some $x$ in $X$ and $\{Ax_n\}$ converges to some $y$ in $Y$, then $y = Ax$. Then $A$ is bounded.*

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**Theorem 5.** *Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a function, and suppose there exists a function $B : \mathcal{H} \longrightarrow \mathcal{H}$ with the property that $\langle Af, g\rangle = \langle f, Bg\rangle$ for all $f$ and $g$ in $\mathcal{H}$. Then $A$ is bounded.*

*Proof.* Let $\{f_n\}$ be a sequence in $\mathcal{H}$ converging to $f$, and suppose that $\{Af_n\}$ converges to $g$. Then for all $h$ in $\mathcal{H}$,

$$\langle g, h\rangle = \lim_{n\to\infty} \langle Af_n, h\rangle = \lim_{n\to\infty} \langle f_n, Bh\rangle = \langle f, Bh\rangle = \langle Af, h\rangle,$$

and therefore $Af = g$. □

Therefore if one wants to study unbounded linear operators on a Hilbert space $\mathcal{H}$ that have an adjoint, such operators can not be defined on all of $\mathcal{H}$.

**Definition 6.** *Let $\mathcal{H}$ be a Hilbert space. An* (unbounded) linear operator *on $\mathcal{H}$ consists of a dense linear subspace $\mathcal{D}(A)$ and a linear map $A : \mathcal{D}(A) \longrightarrow \mathcal{H}$. The linear subspace $\mathcal{D}(A)$ is called the* domain *of $A$.*

**Example 7.** *Let $\phi$ be a continuous unbounded function on $\mathbb{R}$, and define $M^c_\phi$ on $L^2(\mathbb{R})$ by $(M^c_\phi f)(x) = \phi(x)f(x)$ with domain $\mathcal{D}_1$ consisting of compactly supported continuous functions on $\mathbb{R}$.*

**Example 8.** *Let $\phi$ be a continuous unbounded function on $\mathbb{R}$, and define $M_\phi$ on $L^2(\mathbb{R})$ by $(M_\phi f)(x) = \phi(x)f(x)$ with domain*

$$\mathcal{D}_2 = \{f \in L^2(\mathbb{R}) : \phi f \in L^2(\mathbb{R})\}.$$

**Example 9.** *$D$ on $L^2(0, 1)$, defined by $Df = f'$ with domain $C^\infty_0(0, 1)$.*

We shall see that different choices of domain can give a linear operator very different properties.

**Definition 10.** *Let $A$, $B$ be linear operators on $\mathcal{H}$. We say that $A$ and $B$ are* equal *if $\mathcal{D}(A) = \mathcal{D}(B)$ and $Af = Bf$ for all $f$ in $\mathcal{D}(A)$. We will say that $B$ is an* extension *of $A$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Af = Bf$ for all $f$ in $\mathcal{D}(A)$; if $B$ is an extension of $A$, we write $A \subseteq B$.*

Note that in the examples above, the operator $B$ is an extension of $A$.

**Definition 11.** *The* graph *of a linear operator $A$ is the set*

$$\mathcal{G}(A) = \{(f, Tf) : f \in \mathcal{D}(A)\}.$$

Note that if $A \subseteq B$, then $\mathcal{G}(A) \subseteq \mathcal{G}(B)$ as sets.

**Definition 12.** *A linear operator $A$ is* closed *if $\mathcal{G}(A)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$.*

**Theorem 13.** *Let $A$ be a linear operator on $\mathcal{H}$. The following are equivalent:*

(a) *$A$ is closed;*
(b) *If $\{f_n\}$ is a sequence in $\mathcal{D}(A)$ converging to some $f$ in $\mathcal{H}$ and if $\{Af_n\}$ converges to some $g$ in $\mathcal{H}$, then $f$ is in $\mathcal{D}(A)$ and $Af = g$;*
(c) *The linear space $\mathcal{D}(A)$ is a Hilbert space for the inner product*

$$\langle f, g\rangle_A := \langle f, g\rangle + \langle Af, Ag\rangle.$$

**Definition 14.** *A linear operator is* closable *if it has an extension that is closed.*

**Theorem 15.** *Let $A$ be a linear operator on $\mathcal{H}$. The following are equivalent:*

(a) *$A$ is closable;*

(b) *If* $\{f_n\}$ *is a sequence in* $\mathcal{D}(A)$ *converging to* $0$ *and if* $\{Af_n\}$ *converges to some* $g$ *in* $\mathcal{H}$*, then* $g = 0$*;*
(c) *The (set-theoretic) closure of* $\mathcal{G}(A)$ *is the graph of a linear operator* $\bar{A}$*.*

If $A$ is closable, the operator $\bar{A}$ in the previous theorem is unique and is called the *closure* of $A$. The operator $\bar{A}$ is minimal in the following sense: if $B$ is an extension of $A$ that is closed, then $B$ is also an extension of $\bar{A}$.

Our operator $M_\phi$ is closed. The other examples are not closed, but they are closable. In fact, one has to do some amount of work to construct an unbounded operator that is not closable.

**Example 16.** *Choose a linear subspace* $\mathcal{D}$ *of* $\mathcal{H}$*, an unbounded linear functional* $F$ *on* $\mathcal{D}$*, and a vector* $h \neq 0$ *in* $\mathcal{H}$*. Define a linear operator* $A$ *with domain* $\mathcal{D}$ *by the formula* $Tf = F(f)h$*. Because* $F$ *is not continuous at* $0$*, there exists a sequence* $\{f_n\}$ *in* $\mathcal{D}$ *with the property that* $\{f_n\}$ *converges to* $0$*, but* $\{F(f_n)\}$ *does not converge to* $0$*. By passing to a subsequence if necessary, we may assume that there exists a constant* $c > 0$ *such that* $|F(f_n)| \geq c$ *for all natural numbers* $n$*. Define*

$$g_n = \frac{x_n}{F(x_n)}.$$

*Then* $\{g_n\}$ *converges to* $0$*, while*

$$T(g_n) = T\left(\frac{f_n}{F(f_n)}\right) = \frac{1}{F(f_n)}T(f_n) = \frac{1}{F(f_n)}F(f_n)e = e \neq 0.$$

We next discuss adjoints of unbounded operators.

**Definition 17.** *Let* $A$ *be a linear operator on a Hilbert space* $\mathcal{H}$*. Set*

$$\mathcal{D}(A^*) = \{g \in \mathcal{H} : \textit{there exists } h \in \mathcal{H} \textit{ such that } \langle Af, g\rangle = \langle f, h\rangle \textit{ for all } f \in \mathcal{D}(A)\}.$$

*Because* $\mathcal{D}(A)$ *is dense in* $\mathcal{H}$*, the element* $h$*, if it exists, is unique. The operator* $A^*$ *with domain* $\mathcal{D}(A^*)$ *defined by* $A^*h = g$ *is called the* adjoint *of* $A$*.*

By definition, we have the equation

$$\langle Af, g\rangle = \langle f, A^*g\rangle$$

for all $f$ in $\mathcal{D}(A)$ and $g$ in $\mathcal{D}(A^*)$.

It is possible that $\mathcal{D}(A^*)$ is not dense in $\mathcal{H}$. In fact, for our example $T$ above, $\mathcal{D}(A^*) = 0$.

In general, it is very difficult to explicitly determine the elements of $\mathcal{D}(A^*)$, but in certain cases we can do this.

**Theorem 18.** $(M_\phi)^* = M_{\overline{\phi}}$.

*Proof.* Integration by parts shows that $(M_\phi)^*$ is an extension of $M_{\overline{\phi}}$:

$$\langle M_\phi f, g\rangle = \int_{\mathbb{R}} \phi f \overline{g}\, dx = \int_{\mathbb{R}} f \overline{\overline{\phi} g}\, dx = \langle f, M_{\overline{\phi}} g\rangle,$$

and an easy argument with characteristic functions shows that these two operators are equal. $\square$

**Theorem 19.** *Let $A$ be an operator on $\mathcal{H}$.*

(a) *$A$ is closable if and only if $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$.*
(b) *If $A$ is closable, then $(\overline{A})^* = A^*$, and $\overline{A} = A^{**}$.*
(c) *$A$ is closed if and only if $A = A^{**}$.*

We can use this part (a) of this theorem to prove that $i\frac{d}{dx}$ on $(0,1)$ is closable. To come up with a domain for which $i\frac{d}{dx}$ is closed, we begin with a definition. Let us consider another example.

**Definition 20.** *A function $f$ on $[0,1]$ is* absolutely continuous *if there exists a function $h$ in $L^1(0,1)$ such that*

$$f(x) = f(0) + \int_0^x h(t)\,dt$$

*for every $x$ in $[0,1]$.*

Absolutely continuous functions are continuous, and are differentiable almost everywhere; in fact, $f'(x) = h(x)$ almost everywhere. For this reason, we typically write $f'$ for $h$. Let $AC[0,1]$ denote the set of absolutely continuous functions on [01], and define

$$H^1(0,1) = \{f \in AC[0,1] : f' \in L^2(0,1)\}.$$

**Theorem 21.** *Let $D$ be the operator $i\frac{d}{dx}$ on $L^2(0,1)$ with domain $H^1(0,1)$. Then $D$ is closed.*

Before considering more examples, we need the notion of a *multi-index*.

**Definition 22.** *A $d$-dimensional* multi-index *is a $d$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ of nonnegative integers. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.*

**Definition 23.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. For each $d$-dimensional multi-index $\alpha$, set*

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

*and*

$$D^\alpha = (-i)^{|\alpha|} \partial^\alpha.$$

*A $n$-th* order linear partial differential operator *on $\Omega$ is an operator with domain $C_0^\infty(\Omega)$ and formula*

$$L = \sum_{|\alpha| \le n} a_\alpha D^\alpha$$

*for smooth functions $a_\alpha$ in $C^\infty(\Omega)$. The* formal adjoint *of $L$ is the operator*

$$L^+ = \sum_{|\alpha| \le n} D^\alpha \overline{a_\alpha} = \sum_{|\alpha| \le n} a_\alpha^+ D^\alpha.$$

*We say $L$ is* formally self-adjoint *if $L^+ = L$.*

By integrating by parts, we can show that

$$\langle Lf, g\rangle = \langle f, L^+ g\rangle$$

for all $f$ and $g$ in $C_0^\infty(\Omega)$.

We let $L_0$ and $L_0^+$ respectively denote $L$ and $L^+$ with $C_0^\infty(\Omega)$ as its domain. The closure of the graph of $L_0$ and $L_0^+$ determine the *minimal* extensions $L_{min}$ and

$L^+_{min}$. There also exist *maximal* extensions $L_{max}$ and $L^+_{max}$; these are defined using distribution theory. The following equalities hold:

$$(L_0^+)^* = L_{max}, \qquad (L_{max})^* = L_{min}.$$

**Definition 24.** *Let $A$ be a linear operator on $\mathcal{H}$.*

(a) *$A$ is* symmetric *if $A \subseteq A^*$.*
(b) *$A$ is* self-adjoint *if $A = A^*$.*
(c) *$A$ is* essentially self-adjoint *if its closure $\overline{A}$ is self-adjoint.*

The operator $M_\phi$ is self-adjoint if $\phi$ is real-valued, and $i\frac{d}{dx}$ is self-adjoint with the domain $H^1(0,1)$. The relationship between these three notions is complicated in general. For example, as we shall see, there are symmetric operators that are not self-adjoint. To explore this further, we need some terminology.

**Definition 25.** *Let $A$ be a linear operator on $\mathcal{H}$. A complex number $\lambda$ is a* regular point *for $A$ if there exists a positive number $c_\lambda$ with the property that*

$$\|(A - \lambda I)f\| \geq c_\lambda \|f\|$$

*for all $f$ in $\mathcal{D}(A)$. We let $\pi(A)$ denote set of regular points of $A$. For $\lambda$ in $\pi(A)$, we define the* defect number *of $d_\lambda(A)$ of $T$ to be the dimension of the orthogonal complement of the range of $T - \lambda I$.*

**Theorem 26.** *Let $A$ be a linear operator on $\mathcal{H}$.*

(a) *$\pi(A)$ is a nonempty open set in $\mathbb{C}$;*
(b) *A complex number $\lambda$ is in $\pi(A)$ if and only if $(A - \lambda I)$ has a bounded inverse on the range of $A - \lambda I$.*
(c) *If $A$ is closable, then $d_\lambda(A)$ is locally constant.*

The set of regular points of $A$ is related to another quantity:

**Definition 27.** *A complex number $\lambda$ belongs to the* resolvent set *$\rho(A)$ of $A$ if $A - \lambda I$ has a bounded inverse defined on all of $\mathcal{H}$. The complement of the resolvent is called the* spectrum *of $A$ and is denoted $\sigma(A)$.*

**Proposition 28.** $\rho(A) = \{\lambda \in \pi(A) : d_\lambda(A) = 0\}$.

**Proposition 29.** *If $A$ is symmetric, then $\mathbb{C} \backslash \mathbb{R}$ is contained in $\pi(A)$.*

The number $d_\lambda(A)$ is constant for $\Im\lambda > 0$, and also constant for $\Im\lambda < 0$. We shall call these numbers the *deficiency indices* of $A$ and denote them $d_+(A)$ and $d_-(A)$, respectively.

Observe that if $\pi(A)$ contains any real number, then $d_+(A) = d_-(A)$.

Because we are requiring our operators to be densely defined, when $A$ is symmetric we can make alternate definitions

$$d_+(A) = \dim\ker(A^* - iI)$$
$$d_-(A) = \dim\ker(A^* + iI)$$

**Theorem 30.** *Let $A$ be a symmetric closable operator on $\mathcal{H}$.*

(a) *$A$ has a self-adjoint extension if and only if $d_+(A) = d_-(A)$.*
(b) *$A$ is essentially self-adjoint if and only if $d_+(A) = d_-(A) = 0$.*

**Example 31.** *Let $\mathcal{I}$ denote one of these three intervals:*

- $(0,1)$ *(or $(a,b)$ for any $a<b$ in $\mathbb{R}$);*
- $(0,\infty)$*;*
- $\mathbb{R}$.

*Define $A$ by $Af=-if'$ and take the domain of $A$ to be $H^1(\mathcal{I})$. Then $A\subseteq A^*$, so $A$ is symmetric. Furthermore, $g$ is in the kernel of $A^*\mp iI$ is and only if $g$ is in $\mathcal{D}(A^*)$ and $g'=\pm g$ on $\mathcal{I}$.*

*If $\mathcal{I}=(0,1)$, then the kernel of $A^*\mp iI$ is spanned by $e^{\pm x}$. Both of these functions are in $H^1(\mathcal{I})$, so $d_+(A)=d_-(A)=1$.*

*If $\mathcal{I}=(0,\infty)$, then $e^{-x}$ is in $\mathcal{D}(A^*)$, but $e^x$ is not, because its derivative is not in $L^2(0,\infty)$. Therefore $d_+(A)=1$ and $d_-(A)=0$.*

*If $\mathcal{I}=\mathbb{R}$, then neither $e^x$ nor $e^{-x}$ is in $\mathcal{D}(A^*)$ so. Therefore $d_+(A)=d_-(A)=0$.*