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Abstract. The purpose of this talk is to make a connection between Hilbert schemes and foliations. Jouanolou initiated a study of algebraic foliations on complex projective spaces in 1978, classifying some low degree cases, work that was extended later by various authors. I will be talking about more recent work of Correa, Jardim and collaborators towards classifying more general distributions. They have constructed moduli spaces to classify families of algebraic distributions. Moreover, they have given a morphism from these moduli spaces to Hilbert schemes and described the fibers, thereby giving a way to understand distributions as coming from closed subschemes of projective space, which may behave poorly, i.e. fail to be smooth, irreducible, or reduced. Ill start with reminders about how things work in projective algebraic geometry, including the notion of moduli spaces. Then Ill describe the moduli of distributions, the map to Hilbert schemes. Ill try to give concrete examples when possible.

1. Introduction

Recent work of Corréa, Jardim and Muniz has made a connection between distributions on $\mathbb{P}^n$ and Hilbert schemes. A distribution $\mathcal{F}$ on $\mathbb{P}^3$ is a continuous choice of subspaces of the tangent bundle $T_{\mathbb{P}^3}$, which an algebraic geometer will think of as a subsheaf

$$0 \to T_{\mathcal{F}} \to T_{\mathbb{P}^3} \to \mathcal{N}_{\mathcal{F}} \to 0$$

where for technical reasons the cokernel $\mathcal{N}_{\mathcal{F}}$ is a torsion free sheaf. If $\mathcal{N}$ has rank r, then $\mathcal{F}$ is a distribution of codimension r. Given a distribution as above, there is a way to determine a closed subscheme of $\mathbb{P}^3$ called the singular scheme of $\mathcal{F}$, denoted $\operatorname{Sing}(\mathcal{F})$.

$$\mathcal{F} \overset{\Sigma}{\mapsto} \operatorname{Sing}(\mathcal{F})$$

Basic questions about this map are (a) is $\mathcal{F}$ determined by $\operatorname{Sing}(\mathcal{F})$ and (b) given $\operatorname{Sing}(\mathcal{F})$, can we construct $\mathcal{F}$ from this data? Question (a) asks whether Σ is 1-1, (b) asks whether Σ is onto. In fact, the authors construct a moduli space of distributions with fixed Hilbert polynomial and define Σ at the level of functors.

The association above really does use the scheme structure, it is not enough to understand $\operatorname{Sing}(\mathcal{F})$ as a set of points in $\mathbb{P}^3$. My goals in this sequence of talks are as follows:

(1) State main results and give examples
(2) Explain the Hilbert polynomial and the Hilbert scheme
(3) Explain what moduli spaces are
(4) Recall how schemes work

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(5) Motivate the definition of schemes by recalling algebraic varieties, the most important schemes.

2. Classical complex varieties

The classical complex varieties become the most important schemes in the modern language, so we recall them here.

2.1. **Affine varieties.** We are interested in solutions to polynomial equations $f_i(x_1, \dots, x_n) = 0$, where $f_i \in R = \mathbb{C}[x_1, \dots, x_n]$. Therefore we define an ambient affine space $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$ in which to consider these solutions and define the *zero set* of polynomials f_i to be

$$Z(f_i) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f_i(a_1, \dots, a_n) = 0 \text{ for all } i\}$$

Since polynomials are continuous functions, these zero sets are closed in the usual topology. But by themselves they also form the closed sets of a much coarser topology called the *Zariski topology*. It is an easy exercise to check that it is a topology since

(1) $Z(1) = \emptyset$
(2) $Z(0) = \mathbb{A}^n$
(3) $Z(f_i) \cup Z(g_j) = Z(f_i \cdot g_j)$
(4) $\cap_k Z(f_i^k) = Z(\sum_k f_i^k)$

The coarseness of the Zariski topology is illustrated by two startling facts: (a) every non-empty open subset of $\mathbb{A}^n$ is dense and (b) $\mathbb{A}^n$ is compact in the Zariski topology. To see the compactness, suppose U_i is an open cover of $\mathbb{A}^n$. Then $U_i = \mathbb{A}^n - Z(J_i)$ for some ideal J_i. Since the U_i cover $\mathbb{A}^n$, the closed sets $Z(J_i)$ have an empty intersection. By part (4) of the exercise above, this means that $\sum J_i$ is the unit ideal, but then we can find $f_i \in J_i$ so that $\sum f_i = 1$, so that there is a finite number of the J_i whose sum is the unit ideal, when the corresponding finite set of open sets U_i form a subcover.

Given any subset $X \subset \mathbb{A}^n$, we define the *Ideal of* X to by $I(X) = \{f \in R : f(a_1, \dots, a_n) = 0 \text{ for each } (a_1, \dots, a_n) \in X\}$. It is an ideal in the ring R. While it is an infinite set, the ideal structure is helpful: (a) Given any set of polynomials f_i, $Z(f_i) = Z(I)$, where $I = (f_i)$ is the ideal generated by the f_i, so instead of working with arbitrary sets of ideals, we need only look at sets that form ideals and (b) By Hilbert's basis theorem, any ideal $I \subset R$ is finitely generated, so we need only consider a *finite* set of polynomials.

The Hilbert Nullstellensatz explains the connection between zero sets and ideals.

Theorem 2.1. (Hilbert) *Let $X \subset \mathbb{A}^n$ be a zero set and $J \subset R$ an ideal. Then*

(1) $Z(I(X)) = X$.
(2) $I(Z(J)) = \sqrt{J} = \{f \in R : f^p \in J \text{ for some } p > 0\}$.

As a consequence, there is a 1-1 order reversing correspondence between Zariski closed subsets of $\mathbb{A}^n$ and radial ideals of R (an ideal J is radical if $\sqrt{J} = J$).

Example 2.2. For $n = 2$ and variables x, y, consider the polynomials $(y - x)^{50}, (y - x^2)^{50}, (y - x^3)^{50}, \dots$. This is an infinite set of polynomials which generate an ideal $J \subset R$, which is finitely generated (not obvious how!). It would take some work to produce a finite generating set for J, but what is EASY to do is find the corresponding zero set $Z(J)$. The common zeroes of the first two polynomials consist of $(0,0)$ and $(1,1)$, and these happen to be common zeroes to all the polynomials. Therefore $Z(J) = \{(0,0),(1,1)\}$. It is also clear enough that $I(Z(J)) = (y-x, y-x^2)$. Moreover, this ideal is radical, so by Hilbert's Nullstelensatz, $\sqrt{J} = (y-x, y-x^2)$.

A closed set $X = Z(I)$ is *reducible* if it is a union of two proper closed subsets; otherwise X is *irreducible*. An affine variety is a closed irreducible subset $X \subset \mathbb{A}^n$. The *affine coordinate ring* of X is $A(X) = \mathbb{C}[x_1,\dots,x_n]/I(X)$.

Example 2.3. In $\mathbb{A}^2$, the set $X = Z(xy) = Z(x) \cup Z(y)$ is the union of the two coordinate axes, hence is reducible. On the other hand, each coordinate axis is irreducible. The affine coordinate ring of the x-axis is $\mathbb{C}[x,y]/(y) \cong \mathbb{C}[x]$.

Proposition 2.4. *A closed set $X \subset \mathbb{A}^n$ is a variety $\iff$ $I(X)$ is a prime ideal $\iff$ $A(X)$ is an integral domain.*

Example 2.5. Two more examples.

(a) $X = Z(x_2 - x_1^2)$ is a variety in $\mathbb{A}^2$. It is irreducible because $x_2 - x_2^2$ is an irreducible polynomial, hence $(x_2 - x_1^2)$ is a prime ideal. It's affine coordinate ring is $A(X) = \mathbb{C}[x_1,x_2]/(x_2 - x_1^2) \cong \mathbb{C}[x_1]$.

(b) Consider the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $\phi(t) = (t, t^2, t^3)$. The image $Y \subset \mathbb{A}^3$ is a variety, for one can also describe the image points by the ideal $(x_2 - x_1^2, x_3 - x_1^3)$. This ideal is prime because the natural inclusion $\psi : \mathbb{C}[x_1] \hookrightarrow \mathbb{C}[x_1,x_2,x_3]/(x_2 - x_1^2, x_3 - x_1^3)$ is surjective - the reason being that with the relations given in the ideal, any polynomial can be written in terms of x_1 only, one can replace x_2, x_3 with x_1^2, x_1^3 respectively. Since ψ is a ring isomorphism and $\mathbb{C}[x_1]$ is an integral domain, so is $A(Y) \cong \mathbb{C}[x_1]$.

Remark 2.6. Algebraic varieties are building blocks for the (Zariski) closed sets in the sense that each closed $X \subset \mathbb{A}^n$ can be written uniquely as a union of varieties V_i satisfying $V_i \not\subset V_j$ for each $i \neq j$. There is a companion theory of decomposition of radical ideals in $R = \mathbb{C}[x_1,\dots,x_n]$ saying that each radical ideal J can be uniquely written as an intersection of prime ideals p_i with $p_i \not\subset p_j$ for $i \neq j$.

2.2. **Projective varieties.** Projective spaces are defined as $\mathbb{P}^n_{\mathbb{C}} = \mathbb{A}^{n+1} - (0,0,\dots,0)/\sim$ where $(a_0,\dots,a_n) \sim (b_0,\dots,b_n)$ if there is $\lambda \in \mathbb{C}^*$ for which $(a_0,\dots,a_n) = \lambda(b_1,\dots,b_n)$. We wish to look at zero sets of polynomials, but this is not defined unless we use *homogeneous* polynomials $f \in S = \mathbb{C}[x_0,\dots,x_n]$. A polynomial f is homogeneous of degree d if it is a linear combination of monomials of total degree d, or equivalently if $f(\lambda\overline{x}) = \lambda^d f(\overline{x})$ for all $\overline{x} \in \mathbb{A}^{n+1}$. A *homogeneous ideal* $I \subset S$ is one that can be generated by homogeneous polynomials:

Now repeat what we did in the previous section:

(1) Define zero sets $Z(f_i) \subset \mathbb{P}^n$ of a family of homogeneous polynomials f_i.
(2) We may take the f_i to be a finite set because every *homogeneous ideal* $I \subset S$ has a finite generating set.
(3) Put the Zariski topology on $\mathbb{P}^n$.

(4) Hilbert's Nullstellensatz can be modified to give a bijection between zero sets and radical homogeneous ideals $I \subset S$, **except** for the *irrelevant maximal ideal* $(x_0, x_1, \ldots, x_n)$ whose zero set does not correspond to any point in $\mathbb{P}^n$.

(5) $S = \mathbb{C}[x_0, \ldots, x_n]$ is the homogeneous coordinate ring for $\mathbb{P}^n$, but now we view it as a graded ring

$$S = \bigoplus_{d \geq 0} S_d$$

where S_d is the vector space of homogeneous forms of degree d.

(6) A projective variety is a closed irreducible $Y \subset \mathbb{P}^n$ and its homogeneous coordinate ring is $S(Y) = \mathbb{C}[x_0, \ldots, x_n]/I(Y)$, which we now view as a graded ring.

(7) A closed set $Y \subset \mathbb{P}^n$ is a variety $\iff$ $I(Y)$ is a homogeneous prime ideal, or equivalently $S(Y)$ is a graded integral domain.

Remark 2.7. Projective space has a standard open cover by affine spaces. The Zariski closed set $Z(x_0)$ is really the same as the space $\mathbb{P}^{n-1}$ obtained by ignoring one variable and the open compliment $U_{x_0} = \mathbb{P}^n - Z(x_0)$ is homeomorphic to $\mathbb{A}^n$ with its Zariski topology: we define a map $\phi : U_{x_0} \to \mathbb{A}^n$ and its inverse by

$$\phi(x_0, \ldots x_n) = (x_1/x_0, \ldots, x_n/x_0)$$

$$\phi^{-1}(y_1, \ldots, y_n) = (1, y_1, \ldots, y_n)$$

It's easy to check that these are two-sided inverses, giving a bijection of sets. For continuity, notice that for $f \in \mathbb{C}[y_1, \ldots, , y_n]$ of degree d, the polynomial $F(y_0, \ldots, y_n) = y_0^d f(y_1/y_0, \ldots, y_n/y_0)$ is homogeneous of degree d and $\phi^{-1}(Z(f)) = Z(F)$, showing continuity of ϕ. Similarly if F is homogeneous of degree d, we can put $f(y_1, \ldots, y_n) = F(1, y_1, \ldots, y_n)$ and observe that $\phi(Z(F) \cap \mathbb{A}^n) = Z(f)$, establishing bicontinuity. If we do the analogous construction to build U_{x_i}, then $\mathbb{P}^n = \cup_i U_{x_i}$ is an open affine cover of projective space.

Example 2.8. We projectivize the previous example using the open affine cover.

(a) Using the recipe above, the conic $Z(x_2 - x_1^2) \subset \mathbb{A}^2$ projectivizes to a projective conic $Z(x_0 x_2 - x_1^2) \subset \mathbb{P}^2$, it is the Zariski closure of the affine conic via $U_{x_0} \subset \mathbb{P}^2$.

(b) The closure recipe doesn't work as simply as in part (a) in general, simply homogenizing the ideal doesn't always give the closure. To illustrate this, consider Example 2.3 (b). If we projectivize the ideal $(x_2 - x_1^2, x_3 - x_1^3)$ we get $(x_0 x_2 - x_1^2, x_0^2 x_3 - x_1^3)$, but this ideal defines a zero set that is too large. So instead I will projectivize the map $\mathbb{A}^1 \to \mathbb{A}^3$ as $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ by

$$\phi(s, t) = (s^3, s^2 t, s t^2, t^3)$$

when we can spot three generators for the ideal of the image of ϕ, namely

$$(x_0 x_2 - x_1^2, x_0 x_3 - x_1 x_2, x_1 x_3 - x_2^2).$$

This turns out to be the ideal defining the closure of a famous example, the twisted cubic curve, which serves as a counterexample to various naive statements about projective varieties. For example, it requires the three global equations you see, but on each open affine it requires only two equations.

2.3. **The Hilbert polynomial.** One of the big advantages of looking at projective varieties is that there are discrete invariants with which to measure them. This allows one to make sure families of varieties are small enough to find moduli spaces for, we will later see the Hilbert scheme as an example of this. This also allows for the definition of Chern classes of vector bundles. If you are thinking about manifolds, they locally look like $\mathbb{R}^n$, but the manifold $\mathbb{R}^n$ is not globally interesting, it has boring homology groups because it's contractible, it doesn't have any interesting vector bundles, they are all trivial. By working with projective objects, matters are decidedly nontrivial. Here I'll introduce an example of such discrete invariants in the Hilbert polynomial.

Let $Y \subset \mathbb{P}^n$ be a projective variety given by homogeneous ideal $I \subset S = \mathbb{C}[x_0, \dots, x_n]$ and homogeneous coordinate ring $S(Y) = S/I$. The *Hilbert function* for Y is

$$H_Y(d) = \dim_{\mathbb{C}} S(Y)_d$$

the dimension of the vector space of degree d forms on Y (by quotienting out by I, two forms become the same if their difference is equal to zero on Y).

Theorem 2.9. *There is a unique polynomial $P_Y(t) \in \mathbb{Q}[t]$ such that $P_Y(d) = H_Y(d)$ for all $d \gg 0$.*

The polynomial $P_Y(t)$ in the theorem is the *Hilbert polynomial* of Y.

The Hilbert polynomial carries topological information about the variety $Y \subset \mathbb{P}^n$. Suppose Y is smooth, so that it is a complex differentiable manifold. Then

(1) $\deg P_Y(t) = \dim_{\mathbb{C}} Y$ the dimension as a complex manifold.
(2) If $\dim Y = 1$ so that Y is a complex curve (i.e. a real surface) and $P_Y(t) = at + b$, then $a = \deg Y$ is the number of intersection points in $Y \cap H$, where $H \subset \mathbb{P}^n$ is a hyperplane.
(3) If $\dim Y = 1$ and $P_Y(t) = at + b$, then $1 - b = g$, the genus of the real surface Y.

Example 2.10. We compute three examples.

(a) An easy example is $Y = \mathbb{P}^3$. The homogeneous coordinate ring is $S = \mathbb{C}[x_0, \dots, x_n]$. The Hilbert function $H_Y(d) = \dim S_d$ is the dimension of the vector space of d-forms in the variables, which has a monomial basis. To count the degree d monomials, imagine d positions for the variables with 3 dividers - the position of the dividers determines the monomial, so we count $\binom{d+3}{3}$ such monomials, giving this as vector space dimension:

d	0	1	2	3	4	5
$\dim S_d$	1	4	10	20	35	56

The Hilbert polynomial is

$$P_Y(t) = \binom{t+3}{3} = \frac{(t+3)(t+2)(t+1)}{6} = \frac{t^3 + 6t^2 + 11t + 6}{6},$$

which shows the need for rational coefficients.

(b) Take $Y \subset \mathbb{P}^2$ the conic with ideal $I = (q = x_0x_2 - x_1^2)$. Notice that the map $S \to I$ given by $f \mapsto f \cdot q$ is an isomorphism, but it shifts the grading by 2 because $\deg q = 2$. In such cases we keep track of the shift in grading by writing $I \cong S(-2)$. Using the exact sequence $0 \to I \to S \to S(Y) \to 0$ and $I \cong S(-2)$ we compute the Hilbert function:

d	0	1	2	3	4	5
$\dim S_d$	1	3	6	10	15	21
$\dim I_d$	0	0	1	3	6	10
$\dim S(Y)_d$	1	3	5	7	9	11

We can see that $P_Y(t) = 2t + 1$ in this example, it agrees with the Hilbert function for all $d \geq 0$, but disagrees for $d < 0$. The Hilbert polynomial tells us that geometrically the degree of Y is $\deg Y = 2$, which is consistent with Bezout's theorem saying that two curves in $\mathbb{P}^2$ of degrees d and e intersect in $d \cdot e$ points. Also Y has genus 0, so $Y \cong \mathbb{P}^1_{\mathbb{C}}$ is a sphere topologically.

(c) For a tougher example, take $Y \subset \mathbb{P}^3$ to be the twisted cubic curve, which has ideal $I = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2)$. Here we will us a resolution for the ideal to help compute the Hilbert function. The three quadric forms generating I give a map $S(-2)^3 \to I$, but the kernel is not zero. In fact, if you look closely at the three quadric equations for I, you can spot two linear relations, which I'll present in a 3×2 matrix M:

$$M = \begin{pmatrix} x_2 & -x_3 \\ -x_1 & x_2 \\ x_0 & -x_1 \end{pmatrix}$$

Multiplication by M gives a complex

$$0 \to S(-3)^2 \xrightarrow{M} S(-2)^3 \to S \to S(Y) \to 0$$

which turns out to be exact. Notice that the 2×2 minors of M are precisely the generators for the ideal I. This is not a coincidence. The Hilbert-Burch theorem theorem says in general that if M is a non-degenerate $r \times (r+1)$ matrix of forms (of coherent degrees), then we get a similar exact sequence and the $r \times r$ minors give generators for the corresponding ideal. In any event, we can use the sequence above to compute the Hilbert function:

d	0	1	2	3	4	5
$\dim S_d$	1	4	10	20	35	56
$\dim S(-2)^3_d$	0	0	3	12	30	60
$\dim S(-3)^2_d$	0	0	0	2	8	20
$\dim I_d$	0	0	3	10	22	40
$\dim S(Y)_d$	1	4	7	10	13	16

The Hilbert polynomial is $P_Y(t) = 3t + 1$, so $Y \subset \mathbb{P}^3$ is a rational curve (sphere) of degree 3. Since it is the isomorphic image of $\mathbb{P}^1_{\mathbb{C}} \cong S^2$ from the map in Example 2.8 (b), this is not surprising.

2.4. **Regular functions and morphisms of varieties.** The last topic I need to touch on before explaining scheme theory is the notion of regular functions on a variety and what the maps (morphisms) of complex varieties are. At this point, let us also expand our notion of variety. A quasi-projective variety is an open subset of a projective variety and a quasi-affine variety is an open subset of an affine variety, which we may view as an open subset of $\mathbb{P}^n$ via the open cover shown above. The word *variety* refers to any of these possibilities, i.e. a quasi-projective variety.

Let $Y \subset \mathbb{P}^n$ be a variety and $U \subset Y$ an open subset. A function $f : U \to \mathbb{C}$ is *regular* if it locally given by rational functions on $\mathbb{P}^n$, in other words if for each $p \in U$, there is an open neighborhood $p \in V \subset U$ and $g, h \in S$ homogeneous of the same degree for which $f(q) = g(q)/h(q)$ for each $q \in V$.

Notice that for $U \subset Y$, the set $\mathcal{O}(U)$ of all regular functions $f : U \to \mathbb{C}$ forms a ring, in fact a $\mathbb{C}$-algebra, since it contains the constant functions and is closed under sum and product. For open $U \subset V \subset Y$ there are compatible restriction maps $r : \mathcal{O}(V) \to \mathcal{O}(U)$. Further, given regular functions on an open cover of U that agree on overlaps, they glue together uniquely to give a regular function on U. These observations tell us that the assignment $U \mapsto \mathcal{O}(U)$ defines a sheaf of rings on Y.

Example 2.11. A few examples.

(a) Much like bounded holomorphic functions, we have $\mathcal{O}(\mathbb{P}^n) = \mathbb{C}$. Moreover, for any projective variety $Y \subset \mathbb{P}^n$ the result is the same.

(b) There are many more regular functions on affine varieties. For example, on affine space itself we have $\mathcal{O}(\mathbb{A}^n) = A(\mathbb{A}^n) = \mathbb{C}[x_1, \dots, x_n]$, which makes some sense since these are the global polynomial functions on $\mathbb{A}^n$. To see how the definition plays out, suppose $p(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ is a polynomial. To view this as a rational function on $\mathbb{P}^n$ via the open affine cover in Remark 2.7, write $p(x_1/x_0, \dots, x_n/x_0) = g(x_0, \dots x_n)/h(x_0, \dots, x_n)$ as a quotient of rational functions of the same degree (we may take $h = x_0^{\deg p}$) to see p as a regular function on $U_{x_0} \cong \mathbb{A}^n$.

(c) Expanding on (b), if Y is any affine variety, then $\mathcal{O}(Y) \cong A(Y)$.

(d) If $Y \subset \mathbb{A}^n$ is a variety and $H = Z(f) \subset \mathbb{A}^n$ is a hypersurface defined by a single polynomials f, then $\mathcal{O}(Y - Z(f)) \cong A(Y)_f$, the ring obtain from $A(Y)$ by inverting f, also called the localization of $A(Y)$ at f.

2.5. **Morphisms of varieties.** If X, Y are varieties, a morphism $X \to Y$ consists of a continuous function $\phi : X \to Y$ such that for any regular function $f : U \to \mathbb{C}$ with $U \subset Y$ open, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{C}$ is a regular function on $V = \phi^{-1}(U)$. One of the big challenges for Grothendieck when he defined his theory of schemes was to make the morphisms work properly. For the definition of morphism here, you can see that it heavily relies on the embeddings $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ to define regular functions.

Here are some facts that motivated Grothendieck's definition of a scheme.

(1) if $\phi : X \to Y$ is a morphism of varieties and $U \subset X, V \subset Y$ are open with $\phi(U) \subset V$, then the pull-back map $\phi^* : \mathcal{O}(V) \to \mathcal{O}(U)$ is a ring homomorphism of $\mathbb{C}$-algebras.
(2) Every variety is covered by open affine varieties.
(3) There is a natural map $\psi : \mathrm{Mor(X, Y)} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{A(Y), A(X)})$ given by pull-back, where the first domain is the set of morphisms from X to Y and the codomain is the set of $\mathbb{C}$-algebra homomorphisms of rings. If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine varieties, then ψ is a bijection, meaning that morphisms are entirely determined from the affine coordinate rings.

References

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