

LAPLACE AND DOLBEAULT OPERATORS ON SASAKIAN MANIFOLDS

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Abstract: This is joint work with Georges Habib (Lebanese University and Université de Lorraine CNRS) and Robert Wolak (Jagiellonian University). Sasakian manifolds, the odd-dimensional analogues of Kähler manifolds, carry two natural pairs of first-order Dolbeault-type operators on the full complex of differential forms. On such Sasakian manifolds, we establish Kähler-type identities for these operators, and relate the resulting Dolbeault Laplacians to the Hodge Laplacian. Unlike the Kähler case, where $\Delta = 2\Delta_{\bar{\partial}}$, the relation has extra terms coming from the Reeb flow. Using these formulas and a Lefschetz decomposition, we derive lower and upper eigenvalue estimates for Δ on forms depending on the eigenvalues of the Lie derivative of the Reeb vector field.

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Sasakian manifolds (Sasaki [?], 1960) $\approx$ odd dimensional counterparts of Kähler manifolds.

Physics interest: AdS/CFT correspondence.

Let (M^{2n+1}, g, ξ) be a Sasakian manifold. This means that the vector field ξ is a unit Killing vector field such that the endomorphism $\phi : TM \rightarrow TM$ defined by $\phi(\bullet) := \nabla_\bullet \xi$ satisfies, for $X, Y \in \Gamma(TM)$, the relations

$$\begin{aligned}\phi^2 &= -I + \eta \otimes \xi \\ (\nabla_X \phi)(Y) &= \eta(Y) X - g(X, Y) \xi\end{aligned}$$

where $\eta = \xi^\flat$, ∇ = Levi-Civita connection on M .

Normality condition: $\nabla_X d\eta = 2\eta \wedge X^\flat$.

The foliation by the integral curves of ξ gives rise to a transverse Kähler foliation whose complex structure is the endomorphism ϕ . An equivalent definition for a Sasakian manifold is that the Riemannian cone $(C(M) = (0, \infty) \times M, dt^2 + t^2 g)$ is a Kähler manifold.

$\xi^\flat = \eta$ one form equivalent to ξ
 ie. $\eta(X) = \langle \xi, X \rangle$
 $(d\eta)^n \neq 0$

We now compare facts about the various Laplacians on Kähler manifolds with the corresponding Laplacians on ~~Kähler manifolds~~. *Sasakian mfd's*

Decomposition of differential forms:

Kähler manifolds

$$\Omega^m = \bigoplus_{\substack{r+s=m \\ 0 \leq r,s \leq n}} \Omega^{r,s}$$

$$d = d^{1,0} + d^{0,1}$$

$$\partial = d^{1,0}, \quad \bar{\partial} = d^{0,1}$$

Handwritten notes:
 $\partial = \partial_z \wedge \frac{\partial}{\partial z}$
 $\bar{\partial} = \partial_{\bar{z}} \wedge \frac{\partial}{\partial \bar{z}}$

Sasakian manifolds

$$\Omega^m = \bigoplus_{\substack{r+s+t=m \\ 0 \leq r,s \leq n \\ t \in \{0,1\}}} \Omega^{r,s,t}$$

$$d = d^{1,0,0} + d^{0,1,0} + d^{0,0,1} + d^{1,1,-1}$$

$$\partial = d^{1,0,0}, \quad \bar{\partial} = d^{0,1,0}$$

Handwritten notes:
 ∂_z
 $\partial_{\bar{z}}$
 $d^{0,0,1} = \eta_1 \wedge \eta_2$

Other researchers have considered:

$$\begin{aligned}\partial_C &= d^{1,0,0} + \frac{1}{2}d^{1,1,-1} = \partial + \frac{1}{2}(d\eta \wedge)(\xi \lrcorner), \\ \bar{\partial}_C &= d^{0,1,0} + \frac{1}{2}d^{1,1,-1} = \bar{\partial} + \frac{1}{2}(d\eta \wedge)(\xi \lrcorner).\end{aligned}$$

Besides the Hodge Laplacian $\Delta = \underline{d^*d + dd^*}$ on differential forms, we consider the Dolbeault-Laplace operators $\Delta_\partial, \Delta_{\bar{\partial}}, \Delta_{\partial_C}, \Delta_{\bar{\partial}_C}$ defined by the differentials $\partial, \bar{\partial}, \partial_C, \bar{\partial}_C$.

Kähler manifolds: $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$

Sasakian manifolds:

Theorem 1. *Let (M^{2n+1}, g, ξ) be a Sasakian manifold. The Dolbeault Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ are related by*

$$\Delta_\partial = \Delta_{\bar{\partial}} - 2iH\mathcal{L}_\xi. \quad (1)$$

The Hodge Laplace operator acting on differential forms on M satisfies

$$\Delta = 2\Delta_{\bar{\partial}} - 2iH\mathcal{L}_\xi + 2i\eta \wedge (\bar{\partial}^* - \partial^*) + 2i\xi \lrcorner (\bar{\partial} - \partial) + 4L\Lambda + 4\eta \wedge \xi \lrcorner H - \mathcal{L}_\xi^2. \quad (2)$$

Here, H , Λ and L are operators acting on differential forms that yield an $\mathfrak{sl}(2)$ -representation on differential forms, and $\mathcal{L}_\xi$ is the Lie derivative in the direction of ξ .

$$\begin{aligned}
 L &= \frac{1}{2}d\eta \quad \text{multiplication by Kähler form;} & \Lambda &= L^*; \\
 H &= n - (\text{horizontal degree}); & & \\
 [\Lambda, L] &= H, & [L, H] &= 2L, & [\Lambda, H] &= -2\Lambda.
 \end{aligned}$$

$\neq n - r - s \quad \text{on } \Sigma^{r,s} \subset \mathbb{C}$

Theorem 2. (*Hodge Theorem*) *Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. For $0 \leq r, s \leq n$ and $t \in \{0, 1\}$, let $H_{\bar{\partial}}^{r,s,t}(M)$, respectively $H_{\partial}^{r,s,t}(M)$, be the cohomology groups defined by*

$$H_{\bar{\partial}}^{r,s,t}(M) = \frac{\ker(\bar{\partial} : \Omega^{r,s,t} \rightarrow \Omega^{r,s+1,t})}{\operatorname{im}(\bar{\partial} : \Omega^{r,s-1,t} \rightarrow \Omega^{r,s,t})}, \text{ similarly for } H_{\partial}^{r,s,t}(M).$$

Then the natural map $\Omega^{r,s,t} \rightarrow H_{\bullet}^{r,s,t}(M); \omega \mapsto [\omega]$ induces isomorphisms

$$\begin{aligned} \mathcal{H}_{\bar{\partial}}^{r,s,t} &= \ker(\Delta_{\bar{\partial}}^{r,s,t}) = \ker(\bar{\partial}|_{\Omega^{r,s,t}}) \cap \ker(\bar{\partial}^*|_{\Omega^{r,s,t}}) \cong H_{\bar{\partial}}^{r,s,t}(M), \\ \mathcal{H}_{\partial}^{r,s,t} &= \ker(\Delta_{\partial}^{r,s,t}) = \ker(\partial|_{\Omega^{r,s,t}}) \cap \ker(\partial^*|_{\Omega^{r,s,t}}) \cong H_{\partial}^{r,s,t}(M), \end{aligned}$$

which may be infinite dimensional. There is a countable set of charges $q \in \mathbb{R}$ such that the iq -eigenspace $\Omega^{r,s,t}(q)$ of $\mathcal{L}_{\xi}$ on $\Omega^{r,s,t}$ is nonzero, and the spaces above are Hilbert sums of the spaces

$$\begin{aligned} \mathcal{H}_{\bar{\partial}}^{r,s,t}(q) &= \ker(\Delta_{\bar{\partial}}|_{\Omega^{r,s,t}(q)}) \cong H_{\bar{\partial}}^{r,s,t}(q), \\ \mathcal{H}_{\partial}^{r,s,t}(q) &= \ker(\Delta_{\partial}|_{\Omega^{r,s,t}(q)}) \cong H_{\partial}^{r,s,t}(q), \end{aligned}$$

all of which are finite dimensional.

Lefschetz Decomposition consequences:

Kähler manifold
2n-dim

$$\Sigma^m = \Sigma_P^m \oplus \bigoplus_{1 \leq b \leq \lfloor \frac{m}{2} \rfloor} L^b \Sigma_P^{m-2b}$$

$$\Sigma_P^m = \{ \text{primitive forms of deg } m : \Delta \alpha = 0 \}$$

$$= \{ \alpha \in \Sigma^m : [\Delta, L] = 0, [\Delta, \Lambda] = 0 \}$$

Theorem 3. Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. Let $m \in \{1, \dots, 2n+1\}$. For any $q \in \mathbb{R}$, we have the decomposition

$$\mathcal{H}_{\bar{\partial}}^m(q) = \mathcal{H}_{\bar{\partial}, P}^m(q) \oplus \bigoplus_{1 \leq b \leq \lfloor \frac{m}{2} \rfloor} L^b E_{-2bq}^{m-2b}(\Delta_{\bar{\partial}, P}(q)),$$

where $E_{\lambda}^{m-2b}(\Delta_{\bar{\partial}, P}(q))$ denotes the eigenspace of $\Delta_{\bar{\partial}}$ restricted to primitive forms of degree $m-2b$ corresponding to eigenvalue λ . In particular, for $q > 0$, $\mathcal{H}_{\bar{\partial}}^m(q) = \mathcal{H}_{\bar{\partial}, P}^m(q)$.

If $q < 0$

Eigenvalue estimates:

Theorem 4. *Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. Let $m \in \{0, \dots, n-1\}$. If ω is a nonzero eigenform of $\Delta_{\bar{\partial}}$ restricted to m -forms, that is $\Delta_{\bar{\partial}}\omega = \lambda\omega$ with $\mathcal{L}_{\xi}\omega = iq\omega$, for some $q \leq 0$, then the following estimate*

$$\lambda \geq -q(n-m).$$

holds. Equality is attained if and only if $\lambda = q = 0$, that is, $\omega \in \mathcal{H}_{\bar{\partial}}^m(0)$.

Theorem 5. *Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. Let $m \in \{0, \dots, 2n+1\}$. Then*

- *If $m > n+1$ we have $\mathcal{H}_{\bar{\partial}, P}^m(0) = 0$, and $\mathcal{H}_{\bar{\partial}, P}^{n+1}(0) \subset H^{n+1}(M)$.*
- *If $\mathcal{H}_{\bar{\partial}, P}^m(0)$ is nonzero for some $m < n$, the first positive eigenvalue $\lambda_{1, m+1}$ satisfies*

$$\lambda_{1, m+1} < 4(n - m + 1).$$

Also, we have

$$\lambda_{1, m+2b} < 4(b+1)(n - m - b + 1),$$

for any $0 < b \leq \frac{n-m}{2}$.

Theorem 6. *Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. Let $m \in \{0, \dots, 2n+1\}$. Then,*

- *If $m < n$ and $q < 0$ (resp. $m > n+1$ and $q > 0$), we have $\mathcal{H}_{\bar{\partial}}^m(q) = 0$.*
- *If $m = n$, $q < 0$ and $\mathcal{H}_{\bar{\partial}}^m(q) \neq 0$ (resp. $m = n+1$ and $q > 0$), then q^2 is an eigenvalue of the Hodge Laplacian restricted to m -differential forms.*
- *When $\mathcal{H}_{\bar{\partial}}^m(q)$ is nonzero for some $q > 0$ and $m < n+1$, then the first positive eigenvalue $\lambda_{1,m}$ satisfies*

$$\lambda_{1,m} < q^2 + 2q(n - m + 1),$$

and, for $1 \leq b \leq n - m$,

$$\min(\lambda_{1,m+2b}, \lambda_{1,m+2b+1}) < q^2 + 2q(n - m + 1) + 4(b+1)(n - m - b + 1).$$

Relationships between L and eigenforms of $\Delta_{\bar{\partial}}$ and Δ on Kähler manifolds:

$$\begin{aligned} [L, \Delta] &= 0, & [\bar{\partial}^*, \Delta] &= 0 \\ [\Lambda, \Delta] &= 0, & [\bar{\partial}, \Delta] &= 0 \end{aligned}$$

$$E_{\lambda}^m \xrightarrow{L} E_{\lambda}^{m+2}$$

injective
for $m \leq n-2$
isomorphism
for $m = n-1$

Relationships between L and eigenforms of $\Delta_{\bar{\partial}}$ and Δ on Sasakian manifolds:

Theorem 7. *Let (M^{2n+1}, g, ξ) be a compact Sasakian manifold. Let f be an eigenfunction of the $\bar{\partial}$ -Laplacian associated with an eigenvalue q of the Lie derivative, that is, $\Delta_{\bar{\partial}} f = \lambda f$ and $\xi(f) = iqf$. Let $m \in \{1, \dots, n\}$. Then there exist four linear combinations of the forms $\alpha_m := L^m f$, $\eta \wedge \bar{\partial}^* \alpha_m$, $\eta \wedge \partial^* \alpha_m$, and $\bar{\partial} \partial \alpha_{m-1}$ that are eigenforms of degree $2m$ of the Hodge Laplacian corresponding to eigenvalues*

$$\Theta_0(q, \lambda) = 2\lambda - 4m^2 + 4mn + 2nq + q^2, \quad \text{and}$$

and/or

$$\Theta_{\pm}(q, \lambda) = 2\lambda - 4m^2 + 4mn + 4m + 2nq - 2n + q^2 \pm 2\sqrt{2\lambda + (n + q)^2}.$$

When all of the forms $\bar{\partial}^* \alpha_m$, $\eta \wedge \partial^* \alpha_m$, and $\bar{\partial} \partial \alpha_{m-1}$ are nonzero, $\Theta_0(q, \lambda)$ has multiplicity two, and each $\Theta_{\pm}(q, \lambda)$ has multiplicity one.

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