Holomorphic injectivity and the Hopf map

Problem: Given a map $f : X \to \mathbb{R}^n$ that is a local diffeomorphism, or a map $f : X \to \mathbb{C}^n$ that is a local biholomorphism, or a map $f : X \to \mathbb{A}^n_{\mathbb{C}}$ that is a étale morphism, how do recognize when f is (globally) injective?

Examples: In the early 1900's, Bieberbach and Fatou studied univalent maps from the open unit disk D in the complex plane to $\mathbb{C}$, by looking at the coefficients of the Taylor series (assume $f(0) = 0, f'(0) = 1$). The Bieberbach conjecture is that if the map is globally $1-1$, then $\left|f^{(k)}(0)\right| \leq (k!)\, k$. This was proved by de Branges in 1984, and the converse is known to be false.

The **asymptotic stability conjecture** (Marcus Yamabe Conjecture). Let X be a smooth vector field on $\mathbb{R}^2$, $X(0) = 0$. If the eigenvalues of DX have negative real part for $z \in \mathbb{R}^2$, then the origin is a global attractor. This was partially proved by Olech in 1963: If $X : \mathbb{R}^2 \to \mathbb{R}^2$ is injective, then the conjecture holds. Guttierez proved in 1995, if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeo and $[0, \infty) \cap Spec\ DF(z) = \varnothing$ for all $z \in \mathbb{R}^2$, then F is injective.

The **Jacobian conjecture** (Keller, 1939): if $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map and if $\det(Df) = 1$, then f is injective (and thus bijective with polynomial inverse).

The **Hopf map** is defined as follows. $H : S^{2n-1} \to \mathbb{CP}^{n-1}$ ($u \mapsto$complex line through 0 and u). Can also look at $S^{2n-1} \to \mathbb{RP}^{2n-1} \xrightarrow{\pi} \mathbb{CP}^{n-1}$. Fact: Neither H nor π have continuous sections for $n \geq 2$. The proof: if there were such a section s of H, then $H^2\left(\mathbb{CP}^{n-1}\right) \to H^2\left(S^{2n-1}\right) = 0 \to H^2\left(\mathbb{CP}^{n-1}\right)$ can't be the identity.

Toy example: If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a local biholomorphism for $n \geq 2$, such that for each complex line $\updownarrow \in \mathbb{C}^n$, $F^{-1}(\updownarrow)$ is connected and simply connected. Then F is injective.

Sketch of proof: Suppose $F(p) = F(q) = 0$ with $p \neq q$. Then the preimage of a line $\updownarrow$ through the origin is a connected and simply connected set containing p and q. It is a complete, simply-connected real surface of nonpositive curvature in the standard metric (Griffiths-Harris p. 79). Hadamard proved that there exists a unique geodesic on $F^{-1}(\updownarrow)$ that connects p and q, say starting from p at unit speed. Let $w(\updownarrow)$ be the initial vector of this unique unit speed path. It pushes to a point $v(\updownarrow) = dF(p)\, w(\updownarrow)$. Then $v(\updownarrow) / |v(\updownarrow)|$ gives a continuous section of the Hopf map. Contradiction.

Theorem: (Nollet and Xavier). Let X be a connected, complex manifold with dim $n \geq 2$, $F : X \to \mathbb{C}^n$ a local biholomorphism. If each nonempty preimage of a complex line $\updownarrow$ is a connected rational curve (punctured genus zero surface, locally conformal to $\mathbb{C}$), then F is injective.

Cor: If F is étale or algebraic, this is automatic. So F is injective iff each $F^{-1}(\updownarrow)$ is a connected rational curve.

Sketch of proof: similar picture as in the above. Consider the preimage of a line $\updownarrow$ through the origin is a connected and simply connected set containing p and q and r. Then there is a unique fractional linear transformation such that $p \mapsto 0$, $q \mapsto 1$, $r \mapsto \infty$. Now fix τ a unit vector in $\mathbb{C}$ at 0, and then push back to a vector in $\updownarrow$. Again this gives a continuous section of the Hopf map. Contradiction. Proving this map is continuous is the bulk of the proof.

Next time: Birationality of étale morphism via surgery.